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BILLIARDS IN IDEAL HYPERBOLIC POLYGONS

Simon Castle, Norbert Peyerimhoff
Department of Math. Sciences
Durham University
Durham DH1 3LE, UK

and Karl Friedrich Siburh
Technische Universität Dortmund
Fakultät für Mathematik
44221 Dortmund, Germany

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Abstract. We consider billiard trajectories in ideal hyperbolic polygons and present a conjecture about the minimality of the average length of cyclically related billiard trajectories in regular hyperbolic polygons. We prove this conjecture in particular cases, using geometric and algebraic methods from hyperbolic geometry.

1. **Introduction.** In this article we study billiards in polygons of the hyperbolic plane. Our main model of the hyperbolic plane is the Poincaré unit disk $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$. The polygons $\Pi \subset \mathbb{D}$ under consideration are *ideal*, which means that all vertices of Π lie in the boundary at infinity $\partial\mathbb{D} = \{z \in \mathbb{C} \mid |z| = 1\}$.

A billiard curve $c : \mathbb{R} \to \Pi$ is a piecewise smooth curve, parametrized by arclength and consisting of geodesic arcs which are reflected at the walls of the polygon. To avoid technical difficulties, we do not consider billiard curves starting or ending in vertices of the polygon. Moreover, we identify billiard curves up to (orientation-preserving) reparametrizations, hence leading to the same trajectory. A natural way to decode a billiard trajectory is to capture the order in which it hits the polygonal sides. By enumerating the sides of the polygon counter-clockwise from 1 to k, every billiard trajectory gives rise to a bi-infinite *billiard sequence* $(a_j)_{j\in\mathbb{Z}}$ with $a_j \in \{1, \dots, k\}$. Note that we identify sequences which are just shifts of each other and denote the set of all those (identified) billiard sequences by $\mathcal{S}(\Pi)$. In contrast to Euclidean polygonal billiards, every billiard sequence uniquely determines the corresponding billiard trajectory in the hyperbolic polygon (see Theorem 2.1 below).

Obviously, *periodic* billiard trajectories correspond to *periodic* billiard sequences (a_j). A billiard sequence (a_j) with period n is also denoted by $\mathbf{a} = \overline{a_0, a_1, \dots, a_{n-1}}$ (see Figure 1 for illustration). Let $\mathcal{S}_{per}(\Pi)$ be the set of all periodic sequences in $\mathcal{S}(\Pi)$. We can associate to $\mathbf{a} = \overline{a_0, \dots, a_{n-1}}$ a closed billiard trajectory which starts and ends at the side a_0 and hits the sides of the polygon Π in the order $a_1, \dots, a_{n-1}$. The (finite) hyperbolic length of this closed piecewise geodesic curve is denoted by

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$L(\Pi, \mathbf{a})$. In this geometric interpretation, the shift $\overline{a_1, \dots, a_{n-1}, a_0}$ of $\mathbf{a}$ corresponds to the same closed billiard trajectory with a different start point but of the same finite length. As mentioned above, we identify all shifts in the set of all periodic billiard sequences $\mathcal{S}_{per}(\Pi)$. Note that $\mathbf{a}$ not only represents an element in $\mathcal{S}_{per}(\Pi)$, but also contains the information on its period. Thus, $\mathbf{a} = \overline{a_0, a_1, \dots, a_{n-1}}$ and $\mathbf{b} = \overline{a_0, \dots, a_{n-1}, a_0, \dots, a_{n-1}}$ both represent the same element in $\mathcal{S}_{per}(\Pi)$, but we have $L(\Pi, \mathbf{b}) = 2L(\Pi, \mathbf{a})$.

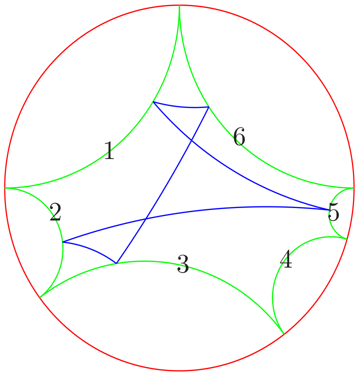


FIGURE 1. Illustration of the periodic trajectory $\overline{1, 5, 2, 3, 6}$

Next we introduce *cyclically related* closed billiard trajectories in a k-gon Π. $\mathbf{a} = \overline{a_0, a_1, \dots, a_{n-1}}$ and $\mathbf{b} = \overline{b_0, b_1, \dots, b_{n-1}}$ are called cyclically related if there is a fixed integer $s \in \mathbb{Z}$ such that

$$b_j \equiv a_j + s \mod k \quad \text{for all } j = 0, 1, \dots, n-1.$$

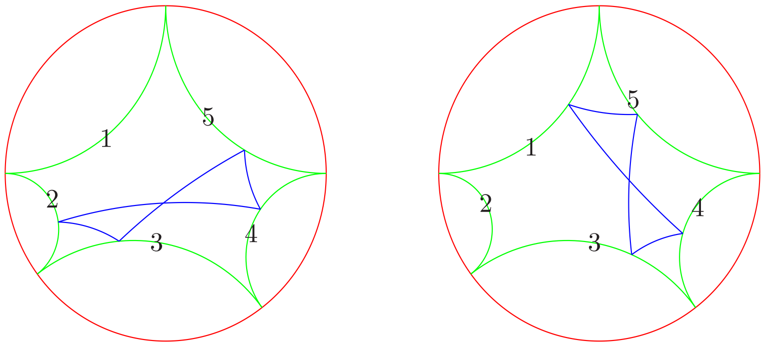
We write $\mathbf{a} \sim \mathbf{b}$, if $\mathbf{a}$ and $\mathbf{b}$ are cyclically related. Another more geometric way to view cyclically related billiard trajectories is to keep the symbolic encoding $\overline{a_0, \dots, a_{n-1}}$, but to change the counter-clockwise enumeration of the sides of Π, i.e., to choose a different side with the label 1. This leads to different closed billiard trajectories in Π which have, however, the same combinatorial structure. Note that "shifts" and "being cyclically related" are completely different concepts: for example, in a pentagon, $\overline{1524}$ and its shifts (e.g., $\overline{5241}$) represent the same periodic billiard sequence, whereas $\overline{1524}$ and its cyclically related sequences (e.g., $\overline{2135}$) are different elements in $\mathcal{S}_{per}(\Pi)$.

Figure 2 presents two cyclically related closed billiard trajectories. The *average length* $L_{av}(\Pi, \mathbf{a})$ is defined as the arithmetic mean of the lengths of all closed billiard trajectories $\mathbf{b}$ which are cyclically related to $\mathbf{a}$, i.e.,

$$L_{av}(\Pi, \mathbf{a}) = \frac{1}{k} \sum_{\mathbf{b} \sim \mathbf{a}} L(\Pi, \mathbf{b}).$$

An ideal k-gon $\Pi \subset \mathbb{D}$ is called *regular* if its symmetry group is the full dihedral group D_k. If Π is regular, we obviously have $L(\Pi, \mathbf{a}) = L(\Pi, \mathbf{b})$ for cyclically related closed trajectories and, therefore, also $L_{av}(\Pi, \mathbf{a}) = L(\Pi, \mathbf{a})$.

We believe that the following statement is true.



(a) Trajectory $\overline{2,3,5,4}$ (b) Trajectory $\overline{3,4,1,5}$

FIGURE 2. Two cyclically related trajectories

Conjecture. *Let $\Pi \subset \mathbb{D}$ be an ideal hyperbolic polygon with k (counter-clockwise enumerated) sides and Π_0 be a regular ideal k-gon (also equipped with an counter-clockwise enumeration of its sides). Let $\mathbf{a} \in \mathcal{S}_{per}(\Pi)$. Then we have*

$$L_{av}(\Pi, \mathbf{a}) \geq L(\Pi_0, \mathbf{a}),$$

with equality if and only if Π is also a regular polygon.

We confirm this conjecture in Sections 4–6 for special billiard trajectories in quadrilaterals, pentagons and hexagons. Our methods of proof are elementary; we use only basic geometric and algebraic techniques in hyperbolic geometry, namely

- hyperbolic trigonometry,
- general Möbius transformations,
- symmetry arguments.

We point out that the algebraic methods prompt a reformulation of our conjecture which is described in Section 3.

It is natural to ask whether regular ideal polygons $\Pi \subset \mathbb{D}$ might have similar *quantum minimality properties*, i.e., whether the bottom $\lambda(\Pi)$ of the Dirichlet spectrum of ideal hyperbolic k-gons is minimal if and only if Π is regular. For hyperbolic quadrilaterals within a *compact* geodesic ball $B \subset \mathbb{D}$, the minimality property of the regular quadrilateral was proved in [17]. In the case of ideal polygons with arbitrarily many sides, it might be possible to deduce this quantum minimality property from our conjecture via a Selberg/Gutzwiller trace formula argument (see [15] for a recommendable and stimulating introduction into quantum mechanics and, in particular, the Gutzwiller trace formula).

Polygonal and polyhedral billiards in hyperbolic space are not only of theoretical interest but appear also in General Relativity in connection with the Mixmaster Universe (see, e.g., the article [19] related to unpublished work of D.M. Chitre) and as Cosmological Billiards (see, e.g., [7]).

The method of studying geodesics on hyperbolic surfaces with the help of symbolic dynamics has a long history and turned out to be very successful with beautiful connections to number theory ([16, 3, 21, 18] are just a few relevant references on this subject).

Other results on billiards in hyperbolic space can be found, e.g., in [4, 8, 9, 13, 14, 11, 24]. Quantum aspects of billiards in the hyperbolic plane are considered, e.g., the articles [1, 12, 20]. For results about dual (or outer) billiards in the hyperbolic plane see, e.g., the survey [23] and the recent article [22].

2. **Billiard sequences.** The following theorem describes the coding of billiard sequences in ideal hyperbolic polygons.

Theorem 2.1. *Let $\Pi \subset \mathbb{D}$ be an ideal polygon with counter-clockwise enumerated sides with labels $1, \dots, k$. A sequence $(a_j) \in \{1, \dots, k\}^{\mathbb{Z}}$ is in $\mathcal{S}(\Pi)$ if, and only if,*

(a) (a_j) does not contain immediate repetitions, i.e., $a_j \neq a_{j+1}$ for all $j \in \mathbb{Z}$.
(b) (a_j) does not contain an infinitely repeated sequence of labels of two adjacent sides.

Moreover, every billiard sequence corresponds to one, and only one, billiard trajectory.

A proof of this theorem was given in the article [10]. However, for the reader's convenience, we present a different proof which is purely geometric in nature. It should also be mentioned that the uniqueness of the coding in Theorem 2.1 does not hold for Euclidean polygons: there are strips of parallel periodic billiard trajectories in Euclidean polygons.

Let us first recall the procedure of *unfolding a billiard trajectory* and fix some notations. Let $\mathbf{a} = (a_j)$ be a billiard sequence in an ideal hyperbolic polygon $\Pi \subset \mathbb{D}$. Let $s_1, \dots, s_k$ denote the sides of Π with the labels $1, \dots, k$, respectively. Instead of reflecting a billiard trajectory (corresponding to $\mathbf{a}$) when it hits the side s_{a_0}, we can continue its direction and reflect the polygon Π at this side. The reflected polygon is denoted by Π' with the sides $s'_1, \dots, s'_k$ (note that the sides s_{a_0} and s'_{a_0} coincide). Repeating this procedure in both future and past, we obtain a single geodesic in $\mathbb{D}$ piercing through a sequence of isometric polygons $\dots, \Pi^{(-2)}, \Pi^{(-1)}, \Pi, \Pi', \Pi'', \Pi^{(3)}, \dots$ at their sides

$$\dots, s^{(-2)}_{a_{-2}}, s^{(-1)}_{a_{-1}}, s_{a_0}, s'_{a_1}, s''_{a_2}, s^{(3)}_{a_3}, \dots.$$

Since Π is an *ideal* polygon, its repeatedly reflected copies $\Pi^{(j)}$ only overlap in the reflection sides. Figure 3 illustrates the first few reflections of the unfolding of the periodic trajectory $\overline{2341}$ in an ideal quadrilateral.

Note also that we can tile all of $\mathbb{D}$ by infinitely many reflected copies of Π. We denote the set of all vertices of these copies by $\mathcal{V}(\Pi) \subset \partial\mathbb{D}$.

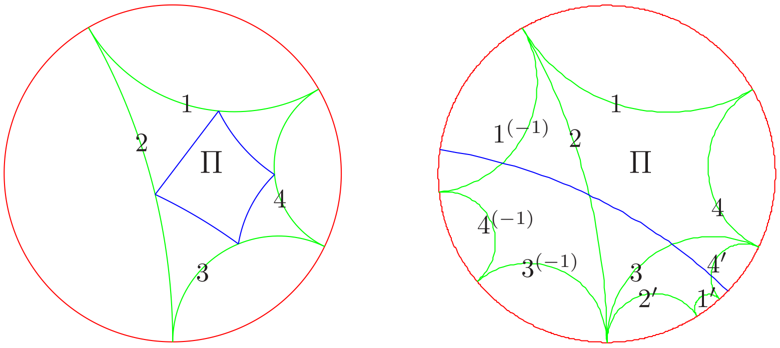
The proof of Theorem 2.1 is based on the following fact in hyperbolic geometry (which follows immediately from the *Nested Intervals Theorem*).

Lemma 2.2. *Let $x \in \mathbb{D}$ be fixed. Every geodesic $c \subset \mathbb{D}$ which does not contain x, divides $\mathbb{D}$ into two open half spaces H^+_c and H^-_c with $x \in H^-_c$. Let $c_0, c_1, c_2, \dots$ be a such sequence of geodesics with the additional properties that*

(i) $c_{j+1} \subset H^+_{c_j}$ for all j and
(ii) $d(c_j, x) \to \infty$.

Then the nested halfspaces $H^+_{c_0} \supset H^+_{c_1} \supset \cdots$ determine a unique ideal limit point $\eta \in \partial\mathbb{D}$ and every geodesic ray connecting a point $y \in H^-_{c_0}$ with η pierces successively once through each of the geodesics $c_0, c_1, \dots$.

Proof of Theorem 2.1. Let us first confirm the necessity of properties (a) and (b) in the theorem. A billiard trajectory cannot hit the same side twice consecutively



(a) Periodic trajectory before unfolding (b) Unfolding and reflected polygons

FIGURE 3. Unfolding a periodic trajectory

since two different geodesics cannot intersect twice. Next we explain the necessity of (b): Let s_j, s_{j+1} (with indices $j, j+1$ taken modulo k) be two adjacent sides of Π. Applying a suitable isometry, we can assume, w.l.o.g, that these two sides are represented as the vertical geodesics $\{x = 0\}$ and $\{x = 1\}$ in the upper half plane model $\mathbb{H} = \{z = x + iy \in \mathbb{C} \mid y > 0\}$. A billiard trajectory not starting or ending in the vertex ∞ of Π must consist of infinitely many arcs of Euclidean semicircles within $\{x + iy \in \mathbb{H} \mid 0 \leq x \leq 1\}$. Assume there is an arc hitting both sides s_j, s_{j+1} successsively. Unfolding this arc yields a full semicircle C and the reflected images of s_j, s_{j+1} are still vertical geodesics at $x = n$, $n \in \mathbb{Z}$, as long as no other sides of Π are hit by the billiard trajectory in the immediate future or past. But the semicircle C can only intersect finitely many of these vertical images, so the forward or backward pattern $j, j+1, j, j+1, \dots$ in the corresponding billiard sequence must be finite.

Now we choose a sequence $\mathbf{a} = (a_j)$ satisfying (a) and (b) and want to construct the corresponding unique billiard trajectory in Π. First choose an arbitrary interior point $x \in \Pi$. The sequence $\mathbf{a}$ tells us how to obtain the sequence of isometric polygons $\Pi^{(j)}$ by reflections in the sides $c_j := s_{a_j}^{(j)}$. The idea then is to use Lemma 2.2 to obtain two limit points, $\alpha \in \partial\mathbb{D}$ associated to the sequence $(c_j)_{j\geq 0}$ and $\omega \in \partial\mathbb{D}$ associated to the sequence $(c_{-j})_{j\geq 1}$. Provided α and ω are different and do not lie in $\mathcal{V}(\Pi)$, the geodesic connecting α with ω will then be the unfolding of the *unique* billiard trajectory corresponding the the billiard sequence $\mathbf{a}$.

The sequence $(c_j)_{j\geq 0}$ obviously satisfies property (i) of Lemma 2.2. Let $d > 0$ denote the minimum of all hyperbolic distances of non-adjacent sides of Π. If a_j and a_{j+1} are the labels of non-adjacent sides, we have

$$d(c_{j+1}, x) \geq d(c_{j+1}, c_j) + d(c_j, x) \geq d + d(c_j, x), \tag{1}$$

since a geodesic from x to a point of c_{j+1} must pass through a point of c_j. From this we can see that if there are infinitely many pairs (a_j, a_{j+1}) (with $j \geq 0$) corresponding to non-adjacent sides, we must have $d(c_j, x) \to \infty$. In this case property (ii) of the lemma is also satisfied and we have a unique limit point $\omega \in \partial\mathbb{D}$.

However, we do not always have an infinite number of pairs (a_j, a_{j+1}) corresponding to non-adjacent sides. (E.g., the billiard sequence $\overline{1234}$ of a quadrilateral has no such pairs.) In this case we consider triplets (a_{j-1}, a_j, a_{j+1}) of sides. If we take the union $\widetilde{\Pi}$ of the two polygons $\Pi^{(j)}$ and $\Pi^{(j+1)}$ at either side of the geodesic c_j, we can treat it as though we are passing through side c_{j-1} in one half of the resulting polygon $\widetilde{\Pi}$, followed by c_{j+1} in the other half. c_{j-1} and c_{j+1} are only adjacent sides of $\widetilde{\Pi}$ if $a_{j-1} = a_{j+1} \equiv a_j \pm 1 \mod k$ (see Figure 4 for a representative example). In the non-adjacent case we have, again, a fixed positive increase of distance similar to inequality (1).

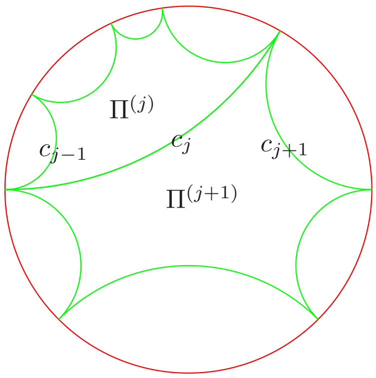


FIGURE 4. Combining two polygons $\Pi^{(j)}$ and $\Pi^{(j+1)}$

Therefore the single case remaining is if there are only finitely many pairs (a_j, a_{j+1}) corresponding to non-adjacent sides and a finite number of triples (a_{j-1}, a_j, a_{j+1}) satisfying $a_{j-1} \neq a_{j+1}$. However, if this were the case, then there would be a $m \geq 0$ such that

$$a_{j-1} = a_{j+1} \equiv a_j \pm 1 \mod k$$

for all $j \geq m$. This would mean that the sequence $a_m, a_{m+1}, \dots$ merely alternates between the labels of two adjacent sides, a situation which we ruled out in the theorem. Therefore, the sequence $(c_j)_{j\geq 0}$ determines a unique limit point $\omega \in \partial\mathbb{D}$. The same argument can be applied to the sequence $(c_{-j})_{j\geq 1}$ (the effective 'past' of the trajectory), giving us again a unique point $\alpha \in \partial\mathbb{D}$. It is clear that the two points α, ω must be distinct: from the above considerations we know that there is a pair (c_j, c_{j+1}) or (c_{j-1}, c_{j+1}) of non-intersecting geodesics with four different end points at infinity. But these geodesics separate completely the future from the past and we must have $\alpha \neq \omega$.

It remains to show that α, ω do not lie in $\mathcal{V}(\Pi)$ (which would lead to a billiard trajectory starting or ending in a vertex of the polygon). But it is easy to see that if $\omega \in \mathcal{V}(\Pi)$ then $(a_j)_{j\geq 0}$ would have eventually to alternate between the labels of two adjacent sides, which is ruled out in the theorem. The same holds true for the limit point α. This finishes the proof of the theorem. $\square$

3. **Reformulation of the conjecture.** In this section we work in the upper half plane model $\mathbb{H}$ with $\partial\mathbb{H} = \mathbb{R} \cup \{\infty\}$. Let $\mathcal{E} \subset \mathbb{R}^{k-1} \subset (\partial\mathbb{H})^{k-1}$ be the set of all $\eta = (\eta_1, \dots, \eta_{k-1})$ with

$$-1 = \eta_1 < \eta_2 < \cdots < \eta_{k-1} = 1.$$

For $\eta \in \mathcal{E}$, we define $\Pi(\eta) \subset \mathbb{H}$ to be the ideal k-gon with the vertices $\eta_1, \dots, \eta_{k-1}$, $\eta_k = \infty$, with an enumeration such that the side $\eta_{l-1}\eta_l$ (with indices taken modulo k) carries the label $l \in \{1, \dots, k\}$. In order to find $\xi = (\xi_1, \dots, \xi_{k-1}) \in \mathcal{E}$ which represents the (unique) regular k-gon in the family $\{\Pi(\eta) \mid \eta \in \mathcal{E}\}$, we consider the regular k-gon in Poincaré unit disk $\mathbb{D}$ with the vertices $e^{\frac{2\pi l}{k} i} \in \partial\mathbb{D}$, $0 \le l \le k-1$. Mapping these vertices back to $\partial\mathbb{H}$ by the Möbius transformation $z \mapsto \frac{i(1+z)}{1-z}$ gives one vertex at ∞ and the other vertices at $-\cot\frac{\pi l}{k}$. Normalising by the factor $\tan\frac{\pi}{k}$ moves these vertices to ∞ and $-1 = \xi_1 < \cdots < \xi_{k-1} = 1$ with

$$\xi_l = -\tan\frac{\pi}{k} \cdot \cot\frac{\pi l}{k} \quad \text{for } 1 \le l \le k-1. \tag{2}$$

The simply transitive action of the general Möbius group on triplets (see, e.g., [2, Section 2.9]) implies that every counter-clockwise enumerated ideal k-gon in $\mathbb{H}$ has a unique isometric image $\Pi(\eta)$ with $\eta \in \mathcal{E}$, such that the enumeration is preserved. Hence, it suffices to consider only the polygons $\Pi(\eta)$ in our conjecture.

Now we give an algebraic description of the hyperbolic reflections along the sides of the polygon $\Pi(\eta)$. Every reflection along a geodesic is of the form $r(z) = \frac{a\bar{z}+b}{c\bar{z}+d}$ with $ad - bc = -1$. For simplicity, we identify the reflection r with the associated matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of determinant -1. Let $r_l(\eta)$ $(l = 1, \dots, k)$ denote the reflection along the side $\eta_{l-1}\eta_l$ of the polygon $\Pi(\eta)$. Then we have

$$r_1(\eta) = \begin{pmatrix} -1 & -2 \\ 0 & 1 \end{pmatrix}, \quad r_k(\eta) = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix} \tag{3}$$

and

$$r_l(\eta) = \frac{1}{\eta_l - \eta_{l-1}} \begin{pmatrix} \eta_{l-1} + \eta_l & -2\eta_{l-1}\eta_l \\ 2 & -(\eta_{l-1} + \eta_l) \end{pmatrix} \quad \text{for } 2 \le l \le k-1. \tag{4}$$

As for the lengths of the closed billiard trajectories in $\Pi(\eta)$, we obtain the following identity.

Proposition 1. *Let $\eta \in \mathcal{E}$ and $\mathbf{a} = \overline{a_0, \dots, a_{n-1}} \in \mathcal{S}_{per}(\Pi(\eta))$. Then*

$$L(\Pi(\eta), \mathbf{a}) = \cosh^{-1}\left(\frac{1}{2}(\text{trace } T_{\mathbf{a}}(\eta))^2 + (-1)^{n-1}\right),$$

where $T_{\mathbf{a}}(\eta) = r_{a_0}(\eta) r_{a_1}(\eta) \cdots r_{a_{n-1}}(\eta)$.

Proof. First, recall that every hyperbolic isometry $f(z) = Tz$ with $T \in SL(2, \mathbb{R})$ fixes two distinct boundary points in $\partial\mathbb{H}$, and the geodesic connecting them is called the *axis* a of f. f translates all points $z \in a$ along the axis by the fixed distance

$$d(T) := d(z, f(z)) = 2\cosh^{-1}\left(\frac{1}{2}|\text{trace } T|\right).$$

$d(T)$ is the so-called *translation length* of f. Obviously, we have $d(T^k) = k\, d(T)$.

To simplify notation, we write $T_{\mathbf{a}}$ instead of $T_{\mathbf{a}}(\eta)$ in this proof. We first consider a billiard sequence $\mathbf{a} = \overline{a_0, \dots, a_{n-1}}$ with n even. Then the corresponding unfolded billiard trajectory agrees with the axis of the hyperbolic isometry $f(z) = T_{\mathbf{a}} z$,

$T_{\mathbf{a}} \in SL(2,\mathbb{R})$ (since n is even), and $L(\Pi(\eta),\mathbf{a})$ coincides with the translation length $d(T_{\mathbf{a}})$. Thus we have

$$L(\Pi(\eta),\mathbf{a}) = 2\cosh^{-1}\left(\frac{1}{2}|\text{trace } T_{\mathbf{a}}|\right). \tag{5}$$

Using the identity trace $(T^2) = (\text{trace } T)^2 - 2 > 0$ for hyperbolic $T \in SL(2,\mathbb{R})$, we derive

$$L(\Pi(\eta),\mathbf{a}) = \frac{1}{2}d(T_{\mathbf{a}}^2) = \cosh^{-1}\left(\frac{1}{2}(\text{trace } T_{\mathbf{a}})^2 - 1\right),$$

proving the proposition in this case.

If n is odd, $T_{\mathbf{a}}$ is a matrix with determinant -1 and we have trace $(T_{\mathbf{a}}^2) = (\text{trace}\, T_{\mathbf{a}})^2 + 2$. Let $\mathbf{b} = \overline{a_0,\ldots,a_{n-1},a_0,\ldots,a_{n-1}}$. Then, applying (5) to $L(\Pi(\eta),\mathbf{b})$, we obtain

$$L(\Pi(\eta),\mathbf{a}) = \frac{1}{2}L(\Pi(\eta),\mathbf{b}) = \cosh^{-1}\left(\frac{1}{2}(\text{trace } T_{\mathbf{a}})^2 + 1\right),$$

finishing the proof of the proposition. □

Our reformulation of the conjecture reads as follows.

Reformulated Conjecture. *Let $\mathbf{a} = \overline{a_0,\ldots,a_{n-1}}$ be a periodic billiard sequence of an ideal hyperbolic k-gon. Then the function $F_{\mathbf{a}} : \mathcal{E} \to \mathbb{R}^+$,*

$$F_{\mathbf{a}}(\eta) = \frac{1}{k}\sum_{\mathbf{b}\sim\mathbf{a}} \cosh^{-1}\left(\frac{1}{2}(\text{trace } T_{\mathbf{b}}(\eta))^2 + (-1)^{n-1}\right) \tag{6}$$

assumes its unique minimum at $\eta = (\xi_1,\ldots,\xi_{k-1})$ with ξ_l given in (2). Recall that

$$T_{\mathbf{b}}(\eta) = r_{b_0}(\eta)\cdots r_{b_{n-1}}(\eta),$$

and $r_l(\eta)$ were defined in (3) and (4).

Remark 1. The above minimization problem becomes particularly simple if all billiard sequences cyclically related to $\mathbf{a} = \overline{a_0,\ldots,a_{n-1}}$ are obtained by cyclic shifts of $a_0,\ldots,a_{n-1}$. For example, this is the case when $\mathbf{a} = \overline{1,2,\ldots,k}$. Then (6) simplifies to

$$F_{\mathbf{a}}(\eta) = \cosh^{-1}\left(\frac{1}{2}(\text{trace } T_{\mathbf{a}}(\eta))^2 + (-1)^{n-1}\right),$$

and straightforward monotonicity arguments yield that it suffices to prove that the function

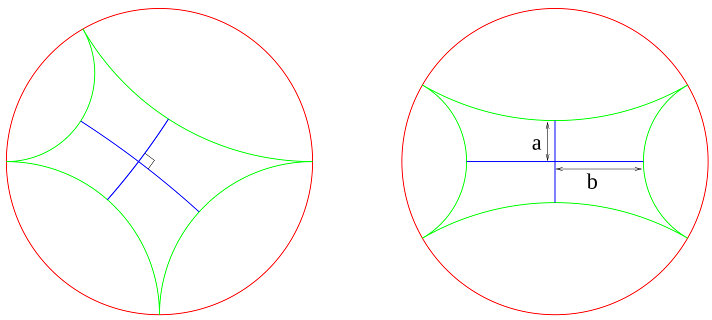
$$G_{\mathbf{a}}(\eta) = |\text{trace } T_{\mathbf{a}}(\eta)| = |\text{trace } (r_{a_0}(\eta)\cdots r_{a_{n-1}}(\eta))|$$

assumes its unique minimum at $\eta = (\xi_1,\ldots,\xi_{k-1})$.

Remark 2. Let us give a short argument which shows, for the basic cyclic sequence $\mathbf{a} = \overline{1,2,\ldots,k}$, the existence of ideal k-gons $\Pi = \Pi(\eta)$ minimizing the length function $L_{av}(\Pi,\mathbf{a}) = L(\Pi,\mathbf{a})$. We need this existence result in Subsection 6.2. Proposition 1 implies that the map $\eta \mapsto L(\Pi(\eta),\mathbf{a})$ is continuous. If the difference $\eta_{l+1} - \eta_l$ tends to zero, then the hyperbolic distance between the side $\eta_l\eta_{l+1}$ and at least one of the sides $\{x = -1\}$ or $\{x = 1\}$ becomes arbitrarily large. This distance is obviously a lower bound for $L(\Pi(\eta),\mathbf{a})$. Therefore, there is a small $\epsilon > 0$ such that $L(\Pi(\eta),\mathbf{a})$ cannot approach its infimum for η outside the compact subset $\mathcal{E}_0 = \{\eta \in \mathcal{E} \mid \eta_{l+1} - \eta_l \geq \epsilon \;\forall\, l\}$. Consequently, the continuous length function must assume a global minimum which lies inside the set $\mathcal{E}_0$.

4. **Quadrilaterals.**

4.1. **Proof for the billiard sequence $\overline{1, 2, 3, 4}$.** We will prove our conjecture for this case by elementary hyperbolic geometry. Consider an ideal quadrilateral Π in the Poincaré unit disk $\mathbb{D}$. The two heights of the pairs of opposite sides are axes of reflective symmetry for Π and meet at a right angle; see Figure 5(a). Without loss of generality, we may assume that we are in the symmetric situation where the intersection point is the centre of the disk, so that the two heights are segments of perpendicular diameters of $\mathbb{D}$, having lengths $2a$ and $2b$, respectively; see Figure 5(b).



(a) The general situation (b) The symmetric situation

FIGURE 5. An ideal quadrilateral in the unit disk model $\mathbb{D}$

It follows that, in this symmetric situation, the geodesic segments through consecutive intersection points of the two heights with each side of Π form a periodic billiard trajectory corresponding to the sequence $\overline{1, 2, 3, 4}$. Each of the four geodesic segments has the same length, say c, so that

$$L_{av}(\Pi, \overline{1, 2, 3, 4}) = 4c.$$

In order to prove our conjecture, we have to show that $c = c(\Pi)$ attains its unique minimum when Π is regular, i.e., when $a = b$.

Note that Π consists of four identical quadrilaterals, each of which has two finite sides of lengths a and b, as well as three right angles and one ideal vertex. It follows that a and b must satisfy

$$\sinh(a) \cdot \sinh(b) = 1 \tag{7}$$

(see, e.g., [5] or [6]). On the other hand, the three segments of lengths a, b and c, respectively, form a right angle triangle which implies that

$$\cosh(c) = \cosh(a) \cdot \cosh(b). \tag{8}$$

Therefore, in view of (7), we obtain

$$\cosh^2(c) = (1 + \sinh^2(a))\Big(1 + \frac{1}{\sinh^2(a)}\Big) = \Big(\sinh(a) + \frac{1}{\sinh(a)}\Big)^2. \tag{9}$$

In order to minimize c for positive values, it is necessary and sufficient to minimize $\cosh(c)$, i.e., the function $u \mapsto u + \frac{1}{u}$ with $u = \sinh(a) > 0$. This function attains its unique minimum at $u = \sinh(a) = 1$, which corresponds precisely to the case of the regular ideal quadrilateral where $a = b$ with $\sinh(a) = 1$, in view of (7).

Thus, the minimal average length of a periodic billiard trajectory corresponding to the sequence $\overline{1, 2, 3, 4}$ is actually attained precisely for the regular ideal quadrilateral, proving our conjecture in this case.

4.2. **Proof for the billiard sequence $\overline{(1,4)^n, 2}$.** Let us now consider the case of periodic billiard trajectories in a quadrilateral corresponding to the sequence $\mathbf{a} = \overline{(1,4)^n, 2}$. For the proof of our conjecture in this situation, we will use its reformulation described in Section 3.

Recalling from Section 3, we denote by $-1 = \eta_1 < \eta_2 < \eta_3 = 1$ the three vertices of the quadrilateral that lie on the real axis; for simplicity, we set $\eta_2 = x$ and $\Pi = \Pi(\eta)$. Then, in view of (3) and (4), the (matrices associated to the) reflections along the sides of the quadrilateral are given by

$$r_1 = \begin{pmatrix} -1 & -2 \\ 0 & 1 \end{pmatrix}, \qquad r_2 = \frac{1}{x+1}\begin{pmatrix} x-1 & 2x \\ 2 & 1-x \end{pmatrix},$$
$$r_3 = \frac{1}{1-x}\begin{pmatrix} 1+x & -2x \\ 2 & -(1+x) \end{pmatrix}, \qquad r_4 = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}.$$

Then we have

$$r_1 r_4 = \begin{pmatrix} 1 & -4 \\ 0 & 1 \end{pmatrix} = \mathbb{1} + N$$

with $N^2 = 0$, which implies

$$(r_1 r_4)^n = (\mathbb{1} + N)^n = \mathbb{1} + nN = \begin{pmatrix} 1 & -4n \\ 0 & 1 \end{pmatrix}.$$

Consequently, the matrix associated to the sequence $\mathbf{a} = \overline{(1,4)^n, 2}$ is given by

$$T_{\mathbf{a}}(\eta) = (r_1 r_4)^n r_2 = \frac{1}{x+1}\begin{pmatrix} x-1-8n & 2x-4n+4nx \\ 2 & 1-x \end{pmatrix}.$$

Since the path $\overline{(1,4)^n, 2}$ has always an odd number of reflections, Proposition 1 tells us that

$$L(\Pi, \mathbf{a}) = \cosh^{-1}\left(\frac{1}{2}(\text{trace } (T_{\mathbf{a}}))^2 + 1\right) = \cosh^{-1}\left(1 + \frac{32n^2}{(1+x)^2}\right). \tag{10}$$

In order to calculate the function $F_{\mathbf{a}}(\eta)$ from (6) we would need to compute the matrices $T_{\mathbf{b}}$ for all four sequences $\mathbf{b} \sim \mathbf{a}$. It is simpler, however, to rotate the quadrilateral such that $\infty \mapsto -1$, $a \mapsto 1$ and $1 \mapsto \infty$, find the new vertex $x' \in (-1, 1)$ (the image of 1 under the rotation), and compute the new matrix $T_{\mathbf{b}}$ by replacing x by x' in (10). Rotating the quadrilateral once yields $x' = -x$, so that rotating twice leads to the vertex x again. Therefore, its average length $F_{\mathbf{a}}(\eta)$, being a function of the real variable x, is given by

$$F(x) = \frac{1}{2}\cosh^{-1}\left(1 + \frac{32n^2}{(1+x)^2}\right) + \frac{1}{2}\cosh^{-1}\left(1 + \frac{32n^2}{(1-x)^2}\right), \tag{11}$$

and we need to determine the minimal value of F over all $x \in (-1, 1)$.

Writing $F(x) = f(x) + f(-x)$ with $f(x) = \frac{1}{2}\cosh^{-1}\left(1 + \frac{32n^2}{(1+x)^2}\right)$, we see that F is an even function so that $F'(x) = 0$ is equivalent to $f'(x) = f'(-x)$. But since f is strictly concave, this implies that $x = 0$. Since $\lim_{x \searrow -1} F(x) = \lim_{x \nearrow 1} F(x) = \infty$, we conclude that the unique quadrilateral of minimal average length is the regular one, proving our conjecture for the case $\mathbf{a} = \overline{(1,4)^n, 2}$.

5. Pentagons.

For the case of pentagons, we use the same ideas as for quadrilaterals developed in Section 4.2. Again, we assume that one vertex of the pentagons Π under consideration is ∞ and the other four are $-1 = \eta_1 < \eta_2 < \eta_3 < \eta_4 = 1$; we call $\eta_2 = x$ and $\eta_3 = y$.

In order to compare to the regular pentagon, we must know what values x and y take for the regular case. Evaluating (2) in the case $k = 5$ yields the values $x = 2 - \sqrt{5}$ and $y = -x = \sqrt{5} - 2$.

In view of (3) and (4), the matrices associated to the reflections along the sides of the pentagon are

$$r_1 = \begin{pmatrix} -1 & -2 \\ 0 & 1 \end{pmatrix}, \; r_2 = \frac{1}{x+1}\begin{pmatrix} x-1 & 2x \\ 2 & 1-x \end{pmatrix}, \quad r_3 = \frac{1}{y-x}\begin{pmatrix} y+x & -2xy \\ 2 & -(y+x) \end{pmatrix}$$
$$r_4 = \frac{1}{1-y}\begin{pmatrix} 1+y & -2y \\ 2 & -(1+y) \end{pmatrix}, \qquad r_5 = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}.$$

5.1. Proof for the billiard sequence $\overline{1,2,3,4,5}$.

Following Remark 1, we just have to prove that the function $G(x,y) = |\mathrm{trace}\ (T_{\mathbf{a}})|$ with $T_{\mathbf{a}} = r_1 r_2 r_3 r_4 r_5$ attains its unique minimum at $x = 2 - \sqrt{5}$ and $y = \sqrt{5} - 2$. Straightforward calculations show that

$$|\mathrm{trace}\ (T_{\mathbf{a}})| = |8f(x,y) - 4| \text{ with } f(x,y) = \frac{1}{y-x} + \frac{2}{(1+x)(1-y)}.$$

It is clear that $(1+x)(1-y) > 0$ and $0 < y - x < 2$, so we have $f(x,y) > \frac{1}{2}$, and hence $8f(x,y) - 4 > 0$; we can therefore drop the modulus signs if we consider $x, y \in (-1,1)$. Setting $\nabla f(x,y) = (0,0)$ is equivalent to the equations

$$(1+x)^2(1-y) = 2(y-x)^2, \tag{12}$$
$$(1+x)(1-y)^2 = 2(y-x)^2. \tag{13}$$

Subtracting (13) from (12), and using $x, y \in (-1,1)$, we conclude that

$$x = -y \tag{14}$$

and, therefore, $2(y-x)^2 = 8x^2$. Hence (12) reads $(1+x)^3 = 8x^2$ with the solutions $x = 1$ and $x = 2 \pm \sqrt{5}$, respectively; the only solution in the interval $(-1,1)$ is given by $x = 2 - \sqrt{5}$ which, in combination with (14), yields the desired result.

5.2. Proof for the billiard sequence $\overline{1,3,5,2,4}$.

For $\mathbf{a} = \overline{1,3,5,2,4}$, we proceed exactly as we did in Section 5.1 for the sequence $\overline{1,2,3,4,5}$. Now the matrix $T_{\mathbf{a}} = r_1 r_3 r_5 r_2 r_4$ satisfies

$$|\mathrm{trace}\ (T_{\mathbf{a}})| = |40f(x,y) - 4|$$

with the same $f(x,y)$ as above. Therefore, the same calculation as in Section 5.1 prove our conjecture also for the case $\mathbf{a} = \overline{1,3,5,2,4}$.

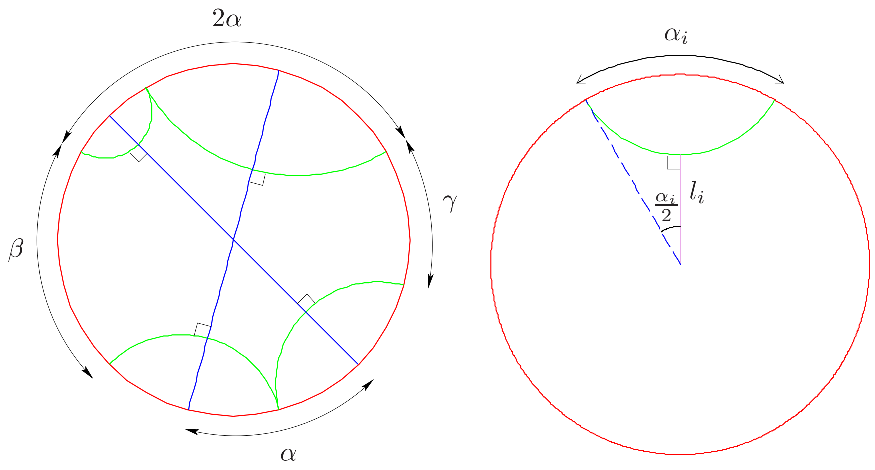
6. **Hexagons and beyond.**

6.1. **Proof for the billiard sequence $\overline{1,4}$.** Note that the length of the billiard trajectory associated to $\overline{1,4}$ in an ideal hyperbolic hexagon is equal to twice the length of the height between the sides 1 and 4. Moreover, we have the following useful fact about the heights.

Proposition 2. *For any ideal hyperbolic hexagon, the three heights meet at a common point.*

The concurrency of the heights holds also for compact right-angled hexagons (see [6, Thm 2.4.3]). However, this is no longer true if we consider ideal $2n$-gons for $n \geq 4$.

Proof. Consider an arbitrary ideal hexagon in $\mathbb{D}$. If we ignore one pair of opposite sides (say 3 and 6) then we are left with four sides defining two heights; these two heights must clearly intersect. By applying a suitable isometry, we can assume that the intersection point is the origin of $\mathbb{D}$, i.e., the two heights are diameters. If α denotes the angle of the two heights between adjacent sides, then the total boundary arc encompassed by either pair of adjacent sides is 2α (see Figure 6(a)). We label the angles of the boundary arcs covered by the sides we dropped (namely, 3 and 6) by β and γ (see, again, Figure 6(a)). The geodesic connecting the midpoints of the two boundary arcs corresponding to the angles β and γ must also be a diameter, since we have $2\alpha + \beta/2 + \gamma/2 = \pi$. By symmetry, this geodesic meets the sides 3 and 6 at right angles, and therefore is an extension of the third height, which also passes through the origin of $\mathbb{D}$. This shows the concurrency of the three heights. $\square$



(a) Two intersecting heights of the hexagon (b) Boundary angle α_i and distance l_i of the side to the origin

FIGURE 6. Notions related to a hexagon in the Poincaré disk

In view of the above proof, it suffices to consider only ideal hyperbolic hexagons with all three heights intersecting in the origin of $\mathbb{D}$. The distance between the origin and one side of the hexagon - say side i - depends only on the angle α_i of the boundary arc covered by this side (see Figure 6(b)). Using hyperbolic trigonometry for right angled triangles, this distance l_i satisfies

$$\cosh l_i = \frac{1}{\sin \frac{\alpha_i}{2}},$$

and, consequently,

$$l_i = \ln\left(\frac{1}{\sin\frac{\alpha_i}{2}} + \sqrt{\frac{1}{\sin^2\frac{\alpha_i}{2}} - 1}\right) = -\ln\tan\frac{\alpha_i}{4}.$$

Note that the length of the height between the sides 1 and 4 is given by $l_1 + l_4$. Our conjecture thus states that the expression

$$\frac{1}{3}(2(l_1+l_4) + 2(l_2+l_5) + 2(l_3+l_6)) = -\frac{2}{3}\ln\left(\prod_{i=1}^{6}\tan\frac{\alpha_i}{4}\right)$$

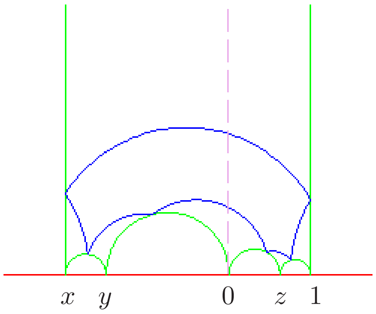
is uniquely minimized in the case of the regular hexagon, i.e., when we have $\alpha_1 = \cdots = \alpha_6 = \frac{\pi}{3}$ (note that the angles α_i always add up to 2π). This leads to the optimisation problem

$$\begin{aligned}&\text{maximize } f(x_1,\ldots,x_6) = \tan x_1 \cdots \tan x_6\\ &\text{subject to } x_1,\ldots,x_6 \geq 0, \sum x_i = \frac{\pi}{2},\end{aligned}$$

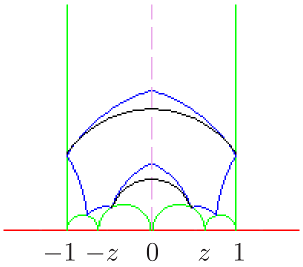
by renaming $\alpha_i/4$ into x_i. We have $f \geq 0$ and, since $\tan x \tan(\frac{\pi}{2} - x) = 1$, f vanishes at all boundary values. This guarantees the existence of a global maximum in the interior. Applying the method of Lagrange multipliers, we conclude that maxima can only lie at points satisfying $x_i \in \{x_1, \frac{\pi}{2} - x_1\}$ for all $i = 2,\ldots,6$. The constraints imply that there is only one maximum at the point $x_1 = \cdots = x_6 = \frac{\pi}{12}$. This proves our conjecture for the billiard sequence $\overline{1,4}$.

6.2. **Proof for the billiard sequence $\overline{1,2,3,4,5,6}$.** Let $\mathbf{a} = \overline{1,2,3,4,5,6}$. We consider ideal hexagons in the upper half plane model $\mathbb{H}$. In contrast to Section 3, we place our hexagons Π (by a suitable isometry) in such a way that two adjacent vertices are $\infty, 1 \in \partial\mathbb{H}$ and the vertex opposite to ∞ is $0 \in \partial\mathbb{H}$. The remaining vertices, denoted by $x, y, z \in \mathbb{R}$, then have to satisfy $x < y < 0 < z < 1$. Assume that Π is such a hexagon and minimizes the length function $L(\Pi, \mathbf{a})$. Its existence is guaranteed by Remark 2. We split Π along the positive imaginary axis into two quadrilaterals (see the dashed line in Figure 7(a)) and consider the length of the billiard trajectory associated to the sequence $\mathbf{a}$ on either side of the split; call these lengths L_- and L_+ for the left and right side, respectively.

We first show that $L_- = L_+$: Let us assume $L_- > L_+$ instead. We then reflect the right-hand side of the hexagon in the dotted line, along with the part of the trajectory in that section. We denote the new symmetric polygon by Π'. The parts of the trajectory joining sides meeting at either 0 or ∞ may no longer form a geodesic segment but, by the triangle inequality, the geodesic segment joining the two points on the sides met by the two fragments will be shorter (see Figure 7(b)), meaning this new closed loop has a length less than or equal to $2L_+ < L_- + L_+$. This may not be a proper billiard trajectory, but its length is certainly an upper bound for the length of the closed billiard trajectory associated to $\mathbf{a}$ in the polygon Π'. Hence, we found another (symmetric) polygon Π' with a smaller $L(\Pi', \mathbf{a})$, which is a contradiction. The case $L_- < L_+$ is treated similary.



(a) Splitting of Π along dashed line



(b) New symmetric polygon Π'

FIGURE 7. Symmetrization of a hexagon and trajectory

Next, we show that the vertices $x < y < 0 < z < 1$ and ∞ of a length minimising hexagon Π satisfy $z = \frac{1}{3}$ and $y = \frac{x}{3}$: From the above considerations we see that the symmetric hexagon Π' with the vertices $-1 < -z < 0 < z < 1$ and ∞ is also length minimising. For this symmetric hexagon, we have the following matrices of reflection:

$$r_1 = \begin{pmatrix} -1 & -2 \\ 0 & 1 \end{pmatrix}, \qquad r_2 = \frac{1}{1-z}\begin{pmatrix} -z-1 & -2z \\ 2 & z+1 \end{pmatrix}, \qquad r_3 = \frac{1}{z}\begin{pmatrix} -z & 0 \\ 2 & z \end{pmatrix}$$
$$r_4 = \frac{1}{z}\begin{pmatrix} z & 0 \\ 2 & -z \end{pmatrix}, \qquad r_5 = \frac{1}{1-z}\begin{pmatrix} 1+z & -2z \\ 2 & -z-1 \end{pmatrix}, \qquad r_6 = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}.$$

To find the value $z \in (0,1)$, for which Π' is length minimising, we have to find the minimum of $|\text{trace } (r_1 \cdots r_6)|$. We obtain

$$|\text{trace } (r_1 \cdots r_6)| = 2 + \frac{16}{z(z-1)^2} = f(z).$$

Since

$$z(z-1)^2 = \frac{4}{27} - \left(\frac{4}{3} - z\right)\left(z - \frac{1}{3}\right)^2 \leq \frac{4}{27},$$

we conclude that $f : (0,1) \to [0,\infty)$ assumes its unique minimum at $z = 1/3$. The proof of $y = \frac{x}{3}$ is similar, but this time we reflect the left-hand side of the original minimising hexagon Π and apply an isometric homothety which sends x and y to -1 and $-y/x$, respectively.

In the last step we show that the length minimising hexagon must satisfy $x = -1$. In view of the previous step, we only have to minimize the length of the cyclic trajectory amongst hexagons with the vertices $x < x/3 < 0 < 1/3 < 1$ and ∞. In this situation, the reflection matrices are given by

$$r_1 = \begin{pmatrix} -1 & 2x \\ 0 & 1 \end{pmatrix}, \qquad r_2 = -\frac{1}{2x}\begin{pmatrix} 4x & -2x^2 \\ 6 & -4x \end{pmatrix}, \qquad r_3 = -\frac{1}{x}\begin{pmatrix} x & 0 \\ 6 & -x \end{pmatrix}$$
$$r_4 = \begin{pmatrix} 1 & 0 \\ 6 & -1 \end{pmatrix}, \qquad r_5 = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}, \qquad r_6 = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix},$$

and we obtain

$$|\text{trace } (r_1 \cdots r_6)| = 56 + 27\left(-x - \frac{1}{x}\right),$$

with $x \in (-\infty, 0)$. The unique minimum in this case is assumed at $x = -1$.

We conclude that the function $L(\Pi, \mathbf{a})$ on hexagons with vertices $x < y < 0 < z < 1$ and ∞ is minimal if and only if the vertices are chosen to be $-1, -1/3, 0, 1/3, 1$ and ∞. Checking with (2), this means that Π is a regular hexagon and the proof is complete.

Finally, we like to mention that a further refinement of the above symmetrization techniques leads to a proof of our conjecture for the cyclic trajectory associated to $\overline{1,2,3,4,5,6,7,8}$ in the octagon. However, we decided to omit this proof, since it does not contain any conceptually new aspects.

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E-mail address: s.c.castle@durham.ac.uk
E-mail address: norbert.peyerimhoff@durham.ac.uk
E-mail address: karl.f.siburg@math.tu-dortmund.de