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Polylogarithms

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Abstract

In this report, I first outline the link between the Dedekind zeta value at 1 and the class number of an algebraic number field in the analytic class number formula. This involves geometrically representing the algebraic number field in a logarithmic space to obtain the Dirichlet regulator, which is explicitly described.
Secondly, I move to the Dedekind zeta value at 2, where we use the Bloch-Wigner function and Bloch group to calculate a 'higher' regulator. This 'higher' regulator is then used in Zagier's conjecture.

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Chapter 1

Introduction

The Polylogarithm is a very simple Taylor series, a generalisation of the widely used logarithm function. The polylogarithm has occurred in situations analagous to other situations involving the logarithm. Naturally, as curious mathematicians, we ask whether the polylogarithm can be found in all places that logarithms appear. This report will cover one example of where we can take an idea using the logarithm 'higher' using the polylogarithm. Polylogarithms can also be found in many other varied areas of mathematics. Searching for interesting links between theories that have common components is one of the main loves of many mathematicians and, in fact, most of the scientific community. Thus, the polylogarithm is of much interest since it can be found in areas as immensely varied as:

- Feynman Integrals.
- As the volume function of hyperbolic tetrahedra.
- Dedekind zeta values for algebraic number fields.
- Regulator functions for algebraic K-groups.

In this report our goal is to obtain evaluations of the Dedekind zeta function using polylogarithms. The first part of the report explains the theory behind the analytic class number formula, which is a fascinating link between the Dedekind zeta value of an algebraic number field at 1 and other important invariants including the Dirichlet regulator and the class number. The calculation of the Dirichlet regulator involves the logarithm of units under a geometric representation of the algebraic number field. As well as allowing us to calculate the Dirichlet regulator, this geometric representation also serves as a very useful way to view the abstract notion of an algebraic number field. We then justify the idea of finding 'higher' Dedekind zeta values with 'higher' polylogarithms and show how this can be done for some specific examples. This will involve utilising the polylogarithm as the volume function of hyperbolic tetrahedra.

This report is aimed at a final year MMATH mathematics student with a background in algebra and number theory but includes some refreshing definitions and reminders to increase accessibility.

For the remainder of this introduction we concern ourselves with defining the polylogarithm, and in particular, defining its domain.

1.1 Defining the Polylogarithm

We start with the well-known natural logarithm,$\ln(x)$, defined over the real numbers, $x \in \mathbb{R}$. A well known result is that the Taylor expansion of $-\ln(1-x)$ over the real numbers is

$$-\ln(1-x) = \sum_{n=1}^{\infty} \frac{x^n}{n} \qquad x \in \mathbb{R},\ |x| < 1.$$

The function extends to the complex plane with $x \in \mathbb{C}$ and $|x| < 1$. It is from this that we get the definition of a polylogarithm.

Definition 1.1.1. *The* ***polylogarithm****, $Li_m(z)$, is defined by a Taylor series as*

$$Li_m(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^m} \qquad z \in \mathbb{C},\ |z| < 1,\ m \in \mathbb{N}.$$

We note that $\mathrm{Li}_1(z) = -\ln(1-x)$. Also note that we are temporarily defining the polylogarithm on the domain $|z| < 1$.

1.1.1 Restricting the Domain

In the above definition we define the polylogarithm with $|z| < 1$. By restricting the polylogarithm to this domain, it is a single valued function. However, as with the natural logarithm, it is possible to extend the domain to a cut complex plane such that the polylogarithm is still single valued. We will see that this cut can be $\mathbb{C} - (1, \infty)$, as seen in figure 1.1.

Let us first examine the domain of the natural logarithm on the complex plane.

Extending the Domain of the Natural Logarithm

We can calculate the natural logarithm of a complex number as $\ln(z) = \ln|z| + i\arg(z), z \in \mathbb{C}$. However, problems arise as the function $\arg(z)$ is not uniquely defined and so neither is $\ln|z| + i\arg(z)$. A simple, if restrictive way to rectify this is to only allow values of $\arg(z)$ to

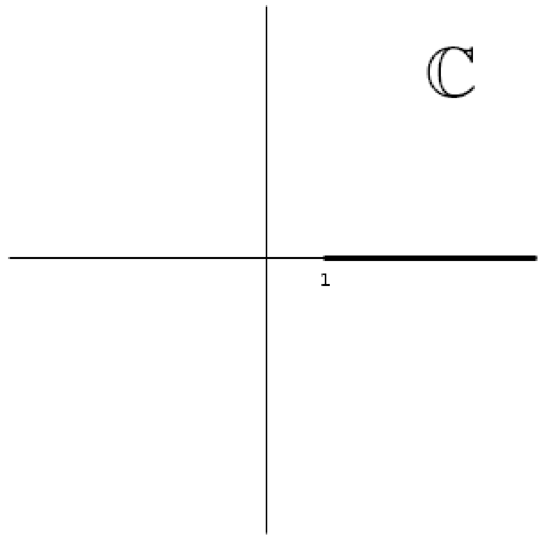

Figure 1.1: The cut plane $\mathbb{C} - (1, \infty)$

be within an open interval of length 2π. In the case where we chose $-\pi < \arg(z) < \pi$, we are simply restricting the function to the cut plane $\mathbb{C} - (-\infty, 0)$

$$\ln(z) = \ln|z| + i\arg(z) \quad z \in \mathbb{C} - (-\infty, 0).$$

We call this the principal branch of the logarithm function.

Next, we will consider $-\ln(1-z)$, the logarithm which coincides with the definition of the polylogarithm at $m = 1$. Using the idea of the principal branch we instead restrict to the cut plane $\mathbb{C} - (1, \infty)$. We do this because as z runs over the interval $(-\infty, 0)$, then $(1 - z)$ runs over $(1, \infty)$. We can now restrict $-\ln(1 - z)$ to get a well defined function,

$$-\ln(1-z) = -\ln|1-z| - i\arg(z) \quad z \in \mathbb{C} - (1, \infty).$$

We can see that $-\ln(1-z)$ is single valued on $|z| < 1$.

Extending the Domain of a Polylogarithm

To calculate the value of a multi-valued function at any point, we calculate the *monodromy* of the function. Monodromy calculates the change in a function as it ‘goes around’ a singularity. For our principal branch of $\ln(z)$ above, this would involve calculating the change in value as the function crossed the cut. The value of $\ln(z)$ would then not only depend on $z \in \mathbb{C}$, but then also on how many times the cut was crossed to reach z. For the logarithm we can calculate the monodromy to be $2\pi i$. In other words, the value of the logarithm of a complex number increases by $2\pi i\mathbb{Z}$ every time the path to reach that point loops anticlockwise around the origin. Calculating the monodromy of a polylogarithm is a rather complicated process and since it does not directly affect the direction of this report we will not discuss it further, the enthusiastic reader is directed to [10].

We can avoid calculating the monodromy for a polylogarithm by finding a restriction of the domain such that the polylogarithm is single valued. We first note the following relation.

Proposition 1.1.2. *Let $z \in \mathbb{C}$ be such that $|z| < 1$, then*

$$\frac{d}{dz} Li_m(z) = \frac{1}{z} Li_{m-1}(z).$$

Proof. Examining the left hand side of the equation, we see that

$$\frac{d}{dz}\mathrm{Li}_m(z) = \frac{d}{dz}\sum_{n=1}^{\infty}\frac{z^n}{n^m}.$$

Next, since this series converges uniformly and absolutely on compact subsets of the unit disk we are able to swap the derivative and the summation.

$$\begin{aligned}
\frac{d}{dz}\sum_{n=1}^{\infty}\frac{z^n}{n^m} &= \sum_{n=1}^{\infty}\frac{d}{dz}\left(\frac{z^n}{n^m}\right) \\
&= \sum_{n=1}^{\infty}\frac{nz^{n-1}}{n^m} \\
&= \sum_{n=1}^{\infty}\frac{z^n}{zn^{m-1}} \\
&= \frac{1}{z}\sum_{n=1}^{\infty}\frac{z^n}{n^{m-1}} \\
&= \frac{1}{z}\mathrm{Li}_{m-1}(z).
\end{aligned}$$

□

We can now extend the domain of the polylogarithm to $\mathbb{C} - (1, \infty)$ using an integral representation derived from the above proposition. By integrating both sides of the relation we can obtain a representation of polylogarithms which will be single valued on the cut plane. For example, for the dilogarithm we obtain

$$\mathrm{Li}_2(z) = -\int_0^z \frac{\ln(1-u)}{u}\,du \qquad \text{for } z \in \mathbb{C} - (1, \infty).$$

The integral representations of ‘higher’ polylogarithms can be found in a similar way.

Now we have an understanding of the polylogarithm, we will actually put it aside until Chapter 5. However, this is not entirely true, since we do use the simplest polylogarithm, the logarithm. The move to the polylogarithm by going ‘higher’, is one of the main ideas in this report. Next, we attempt to geometrically represent an algebraic number field.

Chapter 2

Geometric Representation of an Algebraic Number Field

In this chapter we construct a geometric representation for an algebraic number field and from this give the Dirichlet regulator. As previously mentioned, we first include a brief summary of algebraic number fields and some of their properties. For readers who are fluent in this language, move to section 2.2 for the description of the geometric representation.

2.1 Algebraic Number Fields

2.1.1 Some Definitions

Definition 2.1.1. *Let K be a field extension of L. An element $\alpha \in K$ is called* ***algebraic over*** *L if it satisfies $f(\alpha) = 0$ for some $f(x) \in L[x]$. If all elements in K are algebraic over L then K is called an* ***algebraic extension of*** *L.*

We now narrow the definition to concern only algebraic extensions of $\mathbb{Q}$.

Definition 2.1.2. *A number α which is algebraic over $\mathbb{Q}$ is called an* ***algebraic number****. A field K with $K \supset \mathbb{Q}$ and $[K : \mathbb{Q}] < \infty$ is called an* ***algebraic number field****.*

In this report, we will always denote an algebraic number field by the letter K.

We also have the following essential definition:

Definition 2.1.3. *An* ***algebraic integer*** *is the root of a monic polynomial over $\mathbb{Z}$, i.e α is an algebraic integer if $f(\alpha) = 0$ for some monic $f \in \mathbb{Z}[x]$.*

The algebraic integers in an algebraic number field in fact form a ring. This property is in no way obvious but we will not explain this result here, see pages 91-92 of [4] for a description.

Notation 2.1.4. *The* ***ring of integers*** *of a number field K (sometimes called a* ***number ring****) is denoted $\mathcal{O}_K$.*

2.1.2 Invariants of an Algebraic Number Field

The Discriminant of an Algebraic Number Field

Firstly, the discriminant of an algebraic number field, K, is an important invariant which gives us information about the **ramified primes** in $\mathcal{O}_K$. In a quadratic field (one such as $\mathbb{Q}(\sqrt{d})$, with $1 \neq d \in \mathbb{Z}$ and d squarefree), a prime $p \in \mathbb{Z}$ is ramified in $\mathcal{O}_K$ if $p = \pm\alpha\overline{\alpha}$ where $\alpha \sim \overline{\alpha}$ (meaning that $\alpha = \pm u\overline{\alpha}$ for some unit u, see 2.1.3). The discriminant also in some sense describes the size of $\mathcal{O}_K$ within K.

We can find the discriminant of an n-tuple of elements in K, such as $\underline{\gamma} = (\gamma_1, \ldots, \gamma_n) \in K^n$. We denote the discriminant of this n-tuple $\Delta_K\left(\underline{\gamma}\right)$. To formalise this definition and to define the discriminant d_K of an algebraic number field we must first define the **trace** and **norm** of an element in K.

Definition 2.1.5. *An algebraic number field can be viewed as a vector space over $\mathbb{Q}$. The basis of this vector space will be a set of elements in the algebraic number field. Suppose this basis is $(\beta_1, \ldots, \beta_m)$, we can present a general element α of the algebraic number field in terms of the basis as*

$$\alpha = a_1\beta_1 + \ldots + a_m\beta_m \qquad \textit{with}\, a \in \mathbb{Q},\ \textit{and}\, \beta_i \in K.$$

We can then define a matrix A_α, specific to α, using the following method. We multiply the presentation of α in terms of the basis by each element of the basis in term. We then present each of these m expressions in terms of the basis. In other words we carry out the following m calculations to find the values $a_{ij} \in \mathbb{Q}$ where $1 \leq i, j \leq m$.

$$\begin{aligned} \beta_1(a_1\beta_1 + \ldots + a_m\beta_m) &= a_{11}\beta_1 + \ldots + a_{m1}\beta_m. \\ &\vdots \\ \beta_m(a_1\beta_1 + \ldots + a_m\beta_m) &= a_{m1}\beta_1 + \ldots + a_{mm}\beta_m. \end{aligned}$$

The matrix A_α is defined to be the matrix

$$A_\alpha = (a_{ij})_{1 \leq i,j \leq m,\, i,j \in \mathbb{Z}}.$$

We now define the ***trace of*** *α, denoted $Tr(\alpha)$ to be*

$$Tr(\alpha) = \sum_{i=1}^{m} a_{ii},$$

which is simply the trace of the matrix A_α.

*We also define the **norm of** α, denoted $N(\alpha)$ to be*

$$N(\alpha) = \det\ A.$$

The trace and the norm of an element are independent of the choice of basis.

The norm is a very important function and is used to get an idea of the size of the element. It will play a key role later in the chapter.

We now define formally the discriminant of an n-tuple $\underline{\gamma} = (\gamma_1, \ldots, \gamma_n) \in K^n$.

Definition 2.1.6. *Let $\underline{\gamma} = (\gamma_1, \ldots, \gamma_n) \in K^n$. Then define the matrix $Q\left(\underline{\gamma}\right)$ to be*

$$Q\left(\underline{\gamma}\right) = (Tr(\gamma_i \gamma_j))_{1 \leq i,j \leq m,\, i,j \in \mathbb{Z}}.$$

*We then define the **discriminant** $\Delta_K\left(\underline{\gamma}\right)$ of $\underline{\gamma}$, with respect to K, to be*

$$\Delta_K\left(\underline{\gamma}\right) = \det(Q\left(\underline{\gamma}\right)).$$

*If the n-tuple $\underline{\gamma}$ is in fact a basis of K then this discriminant is the **discriminant of the algebraic number field** K and we denote it d_K. As with the trace and norm, the discriminant of $\underline{\gamma}$ is independent of the choice of basis. d_K is therefore an invariant of the algebraic number field K.*

We demonstrate these definitions in the following example.

Example 2.1.7. *(Trace, norm and discriminant of an algebraic number field) Say we have a field extension $\mathbb{Q}(\theta)$ where θ is the root of the irreducible monic polynomial $f(x) = x^3 - 3x + 4$. In other words $\theta^3 - 3\theta + 4 = 0$.*

We first check this polynomial is irreducible over $\mathbb{Q}$. For this polynomial we can use the rational root theorem to prove irreducibility. This theorem states that any rational root of the polynomial must be of the form $\pm\frac{p}{q}$, where p is an integer factor of the constant term and q is an integer factor of the leading coefficient. For the polynomial $f(x)$, our only possible candidates for roots are $x = \pm 1, \pm 2, \pm 4$. But since

$$f(1) = 2,\ f(-1) = 6,\ f(2) = 6,\ f(-2) = 2,\ f(4) = 56 \text{ and } f(-4) = -48,$$

we deduce that $f(x)$ is irreducible.

We form a basis of $\mathbb{Q}(\theta)$ to be $(\beta_1, \beta_2, \beta_3) = (1, \theta, \theta^2)$ since we have the relation $\theta^3 = 3\theta - 4$.

We then take a general element $\alpha = a + b\theta + c\theta^2 \in \mathbb{Q}(\theta)$ *and multiply it by the elements of the basis in turn and use the identity* $\theta^3 = 3\theta - 4$,

$$\begin{aligned}
\beta_1(a + b\theta + c\theta^2) &= a + b\theta + c\theta^2, \\
\beta_2(a + b\theta + c\theta^2) &= \theta(a + b\theta + c\theta^2) \\
&= a\theta + b\theta^2 + c\theta^3 \\
&= a\theta + b\theta^2 + c(3\theta - 4) \\
&= -4c + (a + 3c)\theta + b\theta^2, \\
\beta_3(a + b\theta + c\theta^2) &= \theta^2(a + b\theta + c\theta^2) \\
&= \theta(-4c + (a + 3c)\theta + b\theta^2) \\
&= -4c\theta + (a + 3c)\theta^2 + b(3\theta - 4) \\
&= -4b + (3b - 4c)\theta + (a + 3c)\theta^2.
\end{aligned}$$

Which forms the matrix

$$A_\alpha = \begin{pmatrix} a & -4c & -4b \\ b & a + 3c & 3b - 4c \\ c & b & a + 3c \end{pmatrix}.$$

The trace of α *is*

$$\begin{aligned}
Tr(a + b\theta + c\theta^2) &= \mathit{Trace}(A_\alpha) \\
&= a + a + 3c + a + 3c = 3a + 6c.
\end{aligned}$$

Also, we have that the norm of this element is

$$\begin{aligned}
N(a + b\theta + c\theta^2) &= \det(A) \\
&= a^3 + 6a^2c + 9ac^2 - 3ab^2 + 12abc - 4b^3 + 12bc^2 + 16c^3.
\end{aligned}$$

We can now find the discriminant of $\mathbb{Q}(\theta)$.

$$\begin{aligned}
d_{\mathbb{Q}(\theta)} = \Delta_{\mathbb{Q}(\theta)}(1, \theta, \theta^2) &= \begin{vmatrix} Tr(1) & Tr(\theta) & Tr(\theta^2) \\ Tr(\theta) & Tr(\theta^2) & Tr(-4 + 3\theta) \\ Tr(\theta^2) & Tr(-4 + 3\theta) & Tr(-4\theta + 3\theta^2) \end{vmatrix} \\
&= \begin{vmatrix} 3 & 0 & 6 \\ 0 & 6 & -12 \\ 6 & -12 & 18 \end{vmatrix} = -324.
\end{aligned}$$

There some nice tricks for calculating d_K in a simpler way than forming the cumbersome matrix above.

Theorem 2.1.8. *Let $K = \mathbb{Q}(\theta)$ and let $p_\theta(x)$ be the minimum polynomial of F. Then*

$$\begin{aligned}
\Delta_{\mathbb{Q}(\theta)}(1, \theta, \dots, \theta^n)) &= \prod_{r>s} (\theta_r - \theta_s)^2 \\
&= (-1)^{\left(\frac{n(n-1)}{2}\right)} \prod_r p_\theta^{'}(\theta_r) \\
&= (-1)^{\left(\frac{n(n-1)}{2}\right)} N(p_\theta^{'}(\theta)).
\end{aligned}$$

Here $p_\theta'(x)$ denotes the derivative of the polynomial $p_\theta(x)$ with respect to x.

Remark 2.1.9. *A few tricks for calculating the norm for $a \in \mathbb{Q}$ and $K = \mathbb{Q}(\theta)$ can be found, such as*

- *$N(a - \theta) = p_\theta(a)$ with θ as above,*
- *$N(a) = a^{[K:\mathbb{Q}]}$.*

With these tools we can now find discriminants very quickly. The following example finds the discriminant from the previous example using these tricks.

Example 2.1.10. *$p_\theta(x) = f(x) = x^3 - 3x + 4$ is our minimum polynomial, we have that $[\mathbb{Q}(\theta) : \mathbb{Q}] = 3$ and $f'(x) = 3x^2 - 3$. So:*

$$\begin{aligned}
f'(\theta) &= 3\theta^2 - 3 \\
&= \frac{3\theta^3 - 3\theta}{\theta} \\
&= \frac{3(3\theta - 4) - 3\theta}{\theta} \\
&= \frac{6(2 - \theta)}{-\theta}, \\
d_F = \Delta_{\mathbb{Q}(\theta)}(1, \theta, \theta^2) &= \frac{(-1)^3 N(6) N(2 - \theta)}{N_F(-\theta)} \\
&= -\frac{6^3 p_\theta(2)}{p_\theta(0)} \\
&= -324.
\end{aligned}$$

Which agrees with our previous calculation of the discriminant.

Real and Complex Places of an Algebraic Number Field

Definition 2.1.11. *We define $r_1 \in \mathbb{Z}^{\geq 0}$ as the number of real places and $r_2 \in \mathbb{Z}^{\geq 0}$ as the number of pairs of complex places of an algebraic number field, K. Each place relates to a root of the minimum polynomial of the extension K over $\mathbb{Q}$. We also have that $[K : \mathbb{Q}] = r_1 + 2r_2$, where $[K : \mathbb{Q}]$ is the degree of K as an extension over the rational numbers.*

This definition is easier to understand after considering the following example.

Example 2.1.12. *Consider $\mathbb{Q}(\theta)$, with θ a root of the polynomial $f(x) = x^3 - 2$. Therefore $\theta = \sqrt[3]{2}$, $\rho\sqrt[3]{2}$ or $\rho^2\sqrt[3]{2}$, where*

$$\rho = \frac{-1 + i\sqrt{3}}{2},$$

the primitive third root of unity. So θ is either a real value, or is, since $\rho^2 = \overline{\rho}$, one of a pair of conjugate complex values, which gives $r_1 = 1$ and $r_2 = 1$. This gives $[\mathbb{Q}(\theta) : \mathbb{Q}] = 3$ as expected.

Class Number

The class number, h_K, of an algebraic number field, K, is an important invariant which is incorporated into one of the main results of this report. We first note the following two definitions.

Definition 2.1.13. *A **fractional ideal of** $\mathcal{O}_K$ is a subset of K of the form*

$$\lambda I = \{\lambda\phi \mid \phi \in I\}$$

where I is a non-zero ideal in $\mathcal{O}_K$ and $\lambda \in K^*$.

We define $J(\mathcal{O}_K)$ to be the set of all fractional ideals in $\mathcal{O}_K$.

Definition 2.1.14. *A fractional ideal of $\mathcal{O}_K$ which is of the form $\lambda\mathcal{O}_K$, where $\lambda \in K^*$, is called a **principal fractional ideal**.*

We define $P(\mathcal{O}_K)$ to be the set of all principal fractional ideals in $\mathcal{O}_K$.

We can now define the class group and the class number of K.

Definition 2.1.15. *The **class group**, $Cl(\mathcal{O}_K)$ is defined to be the quotient group*

$$Cl(\mathcal{O}_K) = \frac{J(\mathcal{O}_K)}{P(\mathcal{O}_K)}.$$

*The order of $Cl(\mathcal{O}_K)$, which is finite, is defined to be the **class number** h_K.*

Intuitively, the class number can be thought to measure the extent unique factorisation into irreducible elements fails in an algebraic number field.

2.1.3 Units of an Algebraic Number Field

We now define the concept of a unit of an algebraic number field. These are, in a sense, elements of 'size' 1 in the algebraic number field. For example, in the trivial algebraic number field $K = \mathbb{Q}$ the units are simply 1 and -1.

Definition 2.1.16. *A* ***unit*** *of an algebraic number field, K, is an element, $u \in \mathcal{O}_K$, such that there exists an element $v \in \mathcal{O}_K$ such that*

$$uv = vu = 1.$$

Remark 2.1.17. *It is very important that u and v are elements in the ring of integers and not just the field, since every element of a field is invertible.*

We also note that $N(u) = \pm 1$. If we multiply any element in K by a unit the norm does not change; this is due to the fact that the norm is multiplicative. So for $x \in K$ and $u \in \mathcal{O}_K$ a unit

$$N(ux) = N(u) \cdot N(x) = \pm N(x).$$

We next quote the following famous theorem by Johann Peter Gustav Lejeune Dirichlet [1], adapted from a version taken from [4]:

Theorem 2.1.18. *For an algebraic number field K of degree $n = r_1 + 2r_2$, there exist units, $\varepsilon_1, \ldots, \varepsilon_r$ with $r = r_1 + r_2 - 1$ such that every unit $u \in \mathcal{O}_K$ has a unique representation in the form:*

$$u = \zeta \varepsilon_1^{a_1} \cdots \varepsilon_r^{a_r}$$

where $a_1, \ldots, a_r \in \mathbb{Z}$ and ζ is some root of unity contained in $\mathcal{O}_K$.

A proof of this theorem can be found on page 112 of [4].

Since every other unit can be essentially obtained as the product of the units $\varepsilon_1, \ldots, \varepsilon_r$ from the above theorem, we call the ε_i a **system of fundamental units**. It is important to note that there is not necessarily a unique system of fundamental units for an algebraic number field.

2.2 Geometric Representation

We now describe how an algebraic number field K can be represented geometrically. This approach to viewing fields at first appears strange, but the relative simplicity and beauty of the representation and the results that arise make it a rather natural way to think about an algebraic number field. A very nice description of this idea appears in [4] (starting from page 113), however we cover this here, based on [4], in a way that most befits our aims.

[1] Born: 13th February 1805, Died: 5th May 1859

2.2.1 The Space Υ^{r_1,r_2}

Firstly we describe a subspace of $\mathbb{C}^{r_1+r_2}$ in which the geometric representations of the elements will exist. We define Υ^{r_1,r_2} to be the space consisting of coordinates of the form

$$(x_1,\ldots,x_{r_1},x_{r_1+1},\ldots,x_{r_1+r_2}) \quad \text{with} \quad \begin{cases} x_i \in \mathbb{R} \text{ if } i=1,\ldots,r_1 \\ x_i \in \mathbb{C} \text{ if } i=r_1+1,\ldots,r_1+r_2. \end{cases}$$

The above vector is an element of $\mathbb{C}^{r_1+r_2}$; that is $\Upsilon^{r_1,r_2} \subset \mathbb{C}^{r_1+r_2}$.

We can also view the size, or **norm** (Note: a different norm to the one previously defined... for now), of an element $x=(x_1,\ldots,x_{r_1+r_2}) \in \Upsilon^{r_1,r_2}$ which I will denote $N_\Upsilon(x)$, by the equation

$$N_\Upsilon(x) = x_1\cdots x_{r_1}\cdot |x_{r_1+1}|^2\cdots|x_{r_1+r_2}|^2.$$

Remark 2.2.1. *The above norm comes from considering Υ^{r_1,r_2} as being within a r_1+2r_2-dimensional space, since the last r_2 coordinates in $(x_1,\ldots,x_{r_1},x_{r_1+1},\ldots,x_{r_1+r_2})$ are complex and can be considered to be a real 2-dimensional space. We can then rewrite the last r_2 coordinates as*

$$x_{r_1+j} = y_{r_1+j} + iz_{r_1+j} \quad \text{for } j=1,\ldots,r_2.$$

This allows us to express x as the vector in $\mathbb{R}^{r_1+2r_2}$,

$$x=(x_1,\ldots,x_{r_1},y_{r_1+1},z_{r_1+1},\ldots,y_{r_1+r_2},z_{r_1+r_2}).$$

We can now view the size of an element x as being the change in size of a general element when multiplied by x, in other words the transformation

$$x' \longrightarrow xx' \quad \text{for } x' \in \Upsilon^{r_1,r_2}.$$

In our original form of coordinates, and for $x'=(x'_1,\ldots,x'_{r_1},y'_{r_1+1}+iz'_{r_1+1},\ldots,y'_{r_1+r_2}+iz'_{r_1+r_2})$ then

$$\begin{aligned} xx' &= (x_1x'_1,\ldots,x_{r_1}x'_{r_1},(y_{r_1+1}+iz_{r_1+1})(y'_{r_1+1}+iz'_{r_1+1}),\ldots,(y_{r_1+r_2},z_{r_1+r_2})(y'_{r_1+r_2},z'_{r_1+r_2})) \\ &= (x_1x'_1,\ldots,x_{r_1}x'_{r_1},y_{r_1+1}y'_{r_1+1}-z_{r_1+1}z'_{r_1+1}+i(y_{r_1+1}z'_{r_1+1}+z_{r_1+1}y'_{r_1+1}),\ldots \\ &\quad \ldots,y_{r_1+r_2}y'_{r_1+r_2}-z_{r_1+r_2}z'_{r_1+r_2}+i(y_{r_1+r_2}z'_{r_1+r_2}+z_{r_1+r_2}y'_{r_1+r_2})). \end{aligned}$$

So when viewed as an element in an r_1+2r_2-dimensional space

$$\begin{aligned} xx' &= (x_1x'_1,\ldots,x_{r_1}x'_{r_1},y_{r_1+1}y'_{r_1+1}-z_{r_1+1}z'_{r_1+1},y_{r_1+1}z'_{r_1+1}+z_{r_1+1}y'_{r_1+1},\ldots \\ &\quad \ldots,y_{r_1+r_2}y'_{r_1+r_2}-z_{r_1+r_2}z'_{r_1+r_2},y_{r_1+r_2}z'_{r_1+r_2}+z_{r_1+r_2}y'_{r_1+r_2}). \end{aligned}$$

We now view the transformation as being a matrix, A, such that

$$x'A = xx'.$$

Here we are viewing x and x' as being in an r_1+2r_2-dimensional space. This gives rise to the matrix

$$A = \begin{pmatrix} x_1 & & & & & & & \\ & \ddots & & & & & & \\ & & x_{r_1} & & & & & \\ & & & y_{r_1+1} & -z_{r_1+1} & & & \\ & & & z_{r_1+1} & y_{r_1+1} & & & \\ & & & & & \ddots & & \\ & & & & & & y_{r_1+r_2} & -z_{r_1+r_2} \\ & & & & & & z_{r_1+r_2} & y_{r_1+r_2} \end{pmatrix}.$$

The scaling of the transformation can be viewed as the determinant of A, given by

$$\begin{aligned} \det A &= x_1 \cdots x_{r_1} \cdot (y_{r_1+1}^2 + z_{r_1+1}^2) \cdots (y_{r_1+r_2}^2 + z_{r_1+r_2}^2) \\ &= x_1 \cdots x_{r_1} \cdot |x_{r_1+1}|^2 \cdots |x_{r_1+r_2}|^2. \end{aligned}$$

This suggests our definition of the norm of an element in Υ^{r_1,r_2}.

2.2.2 Embeddings of the Algebraic Number Field into $\mathbb{C}$

For an algebraic number field, K, there exists exactly n embeddings (which are actually field homomorphisms) into $\mathbb{C}$, where n is the degree of the extension, so $n = [K : \mathbb{Q}]$. We will denote these

$$\sigma_i : K \longmapsto \mathbb{C},\ i = 1, \ldots, n.$$

Each one represents a different embedding of the algebraic number field into the complex numbers, $\mathbb{C}$.

The idea of embedding the algebraic number field into the complex numbers initially sounds like a technicality, since an algebraic number field is constructed by adding to $\mathbb{Q}$ the roots of a polynomial over $\mathbb{Q}$; which are typically identified with elements of $\mathbb{C}$. However, say we have $K = \mathbb{Q}(\theta)$ where θ is the root an irreducible polynomial with irrational roots. Since θ represents all the roots of the polynomial, it is better to think of it as an element in an abstract field, not contained in the complex numbers. To make this clearer, we recall that:

$$\mathbb{Q}(\theta) = \frac{\mathbb{Q}[x]}{p_\theta(x)}$$

where $p_\theta(x)$ is the minimum polynomial of θ over $\mathbb{Q}$.

However, you can view each element of the algebraic number field as an element in the complex numbers via different embeddings of the algebraic number field, one for each root of the polynomial. In a sense, each embeddings replaces θ with one specific root, enabling us to view elements as simply being complex numbers.

For an algebraic number field K, we have the embeddings

$$\sigma_1, \ldots, \sigma_{r_1}, \sigma_{r_1+1}, \overline{\sigma}_{r_1+1}, \ldots, \sigma_{r_1+r_2}, \overline{\sigma}_{r_1+r_2}.$$

The first r_1 embeddings represent those for the real roots of the mimimal polynomial of K, the latter $2r_2$ represent the embeddings for the pairs of conjugate complex roots. Here, we are using the definition

$$\overline{\sigma}_i(\alpha) := \overline{\sigma_i(\alpha)} \quad \alpha \in K.$$

Where we are denoting $\overline{x} \in \mathbb{C}$ to be the complex conjugate of $x \in \mathbb{C}$. The above statement in words tells us that the embedding of the conjugate of a root is the same as the conjugate of the embedding of the root.

We now associate the element $\alpha \in K$ to the element $x(\alpha) \in \Upsilon^{r_1,r_2}$ where

$$x(\alpha) = (\sigma_1(\alpha), \ldots, \sigma_{r_1}(\alpha), \sigma_{r_1+1}(\alpha), \ldots, \sigma_{r_1+r_2}(\alpha)).$$

This is our geometric representation of α. Notice how we only include one of the two possible embeddings for each pair of complex conjugate roots. We now see some useful properties of this representation.

2.2.3 Properties of the Representation

The first result we would like to be true is that this representation is unique for each element within the algebraic number field. This is easy to check since for $\alpha, \beta \in K$ with $\alpha \neq \beta$, then $\sigma_i(\alpha) \neq \sigma_i(\beta)$, for $i = 1, \ldots, n$. Therefore

$$x(\alpha) \neq x(\beta) \quad \alpha, \beta \in K,\ \alpha \neq \beta.$$

We therefore have an injective map

$$\alpha \longmapsto x(\alpha) \quad \alpha \in K.$$

The representation is, of course, not surjective; not every point of $\Upsilon^{r_1+r_2}$ is the image of an element of the algebraic number field.

Proposition 2.2.2. *For $\alpha, \beta \in K$:*

1. $x(\alpha + \beta) = x(\alpha) + x(\beta)$.
2. $x(\alpha)x(\beta) = x(\alpha\beta)$.
3. $x(a\alpha) = ax(\alpha) \quad \forall a \in \mathbb{Q}$.

Proof. 1. Since each σ_i is a homomorphism, $\sigma_i(\alpha+\beta) = \sigma_i(\alpha) + \sigma_i(\beta)$. So

$$\begin{aligned} x(\alpha+\beta) &= (\sigma_1(\alpha+\beta), \ldots, \sigma_{r_1+r_2}(\alpha+\beta)) \\ &= (\sigma_1(\alpha)+\sigma_1(\beta), \ldots, \sigma_{r_1+r_2}(\alpha)+\sigma_{r_1+r_2}(\beta)) \\ &= (\sigma_1(\alpha), \ldots, \sigma_{r_1+r_2}(\alpha)) + (\sigma_1(\beta), \ldots, \sigma_{r_1+r_2}(\beta)) \\ &= x(\alpha) + x(\beta). \end{aligned}$$

2. Since each σ_i is a homomorphism, $\sigma_i(\alpha\beta) = \sigma_i(\alpha)\sigma_i(\beta)$. So

$$\begin{aligned} x(\alpha\beta) &= (\sigma_1(\alpha\beta), \ldots, \sigma_{r_1+r_2}(\alpha\beta)) \\ &= (\sigma_1(\alpha)\sigma_1(\beta), \ldots, \sigma_{r_1+r_2}(\alpha)\sigma_{r_1+r_2}(\beta)) \\ &= (\sigma_1(\alpha), \ldots, \sigma_{r_1+r_2}(\alpha))(\sigma_1(\beta), \ldots, \sigma_{r_1+r_2}(\beta)) \\ &= x(\alpha)x(\beta). \end{aligned}$$

3. For $a \in \mathbb{Q}$, then $\sigma_i(a) = a$, where $i = 1, \ldots, n$. This result is then obtained in a similar way to 2.

□

We have seen what we mean by the norm of an element in $\Upsilon^{r_1+r_2}$. However, we defined earlier what is meant by the norm of an element in an algebraic number field. This leads us to ask the question whether we can relate the norm of α in K and the norm of $x(\alpha)$ in $\Upsilon^{r_1+r_2}$. In a sense we are asking whether the representation retains this information. With some relief, we obtain the following result.

Proposition 2.2.3. *The relation*

$$N_\Upsilon(x(\alpha)) = N(\alpha)$$

holds for $\alpha \in K$.

Proof. The proof of this proposition follows from some basic facts about field extensions. From page 404 of [4] we obtain that the norm of an element in K can be defined by multiplying the different embeddings into $\mathbb{C}$ of the element. Using previous notation, this gives

$$\begin{aligned} N(\alpha) &= \sigma_1(\alpha)\cdots\sigma_{r_1}(\alpha)\cdot\sigma_{r_1+1}(\alpha)\cdot\overline{\sigma}_{r_1+1}(\alpha), \cdots, \sigma_{r_1+r_2}(\alpha)\cdot\overline{\sigma}_{r_1+r_2}(\alpha) \\ &= \sigma_1(\alpha)\cdots\sigma_{r_1}(\alpha)\cdot|\sigma_{r_1+1}(\alpha)|^2\cdots|\sigma_{r_1+r_2}(\alpha)|^2 \\ &= N_\Upsilon(x(\alpha)). \end{aligned}$$

□

We will see this property become very significant when we represent the units of the algebraic number field.

2.2.4 A Motivational Example

The representation we have described has some nice properties. However, with a small modification it can be greatly improved. The modification is simply to apply the logarithm function to the coordinates. Incorporating the properties of the logarithm into the representation really helps us to visualise the algebraic number field; an abstract idea which is incredibly hard to imagine, in a beautiful and astonishing visual form. Using one of the simplest kinds of algebraic number field, a quadratic field, we motivate this change.

Example 2.2.4. *The algebraic number field we will examine here is the field* $K = \mathbb{Q}(\phi)$ *with* $\phi^2 - 5 = 0$. *This has* $r_1 = 2$ *and* $r_2 = 0$. *The two embeddings for this algebraic number field are* σ_1 *and* σ_2, *where*

$$\sigma_1(\phi) = \sqrt{5} \quad \text{and} \quad \sigma_2(\phi) = -\sqrt{5}.$$

We also define a natural norm for the field,

$$N(a + b\sqrt{5}) = a^2 - 5b^2.$$

A fundamental unit for this representation is $u = 2 + \sqrt{5}$ *and using theorem 2.1.18, any unit of the algebraic number field is of the form* $\pm(2 + \sqrt{5})^n$, *for* $n \in \mathbb{Z}$.

We will look at the units

- $u_1 = 2 + \sqrt{5}$,
- $u_2 = (2 + \sqrt{5})^2 = 9 + 4\sqrt{5}$,
- $u_3 = (2 + \sqrt{5})^3 = 38 + 17\sqrt{5}$,
- $u_4 = (2 + \sqrt{5})^4 = 161 + 72\sqrt{5}$,

and

- $u_5 = (2 + \sqrt{5})^5 = 682 + 305\sqrt{5}$.

Under our constructed representation, an element $\alpha \in K$ *is related to a coordinate in* $\mathbb{R}^2$, *namely*

$$(\sigma_1(\alpha), \sigma_2(\alpha)).$$

Figure 2.1 shows a point plot of these coordinates. The points do not tell us much about the elements they are representing. As advertised, we now make the small modification and represent the element as the coordinate

$$(\ln(\sigma_1(\alpha)), \ln(\sigma_2(\alpha))).$$

Figure 2.2 shows the five fundamental units under this new representation. This looks a lot more promising, we can see the five units we have tested appear to lie on a line.

The above example motivates us to explore the addition of logarithms in our representation.

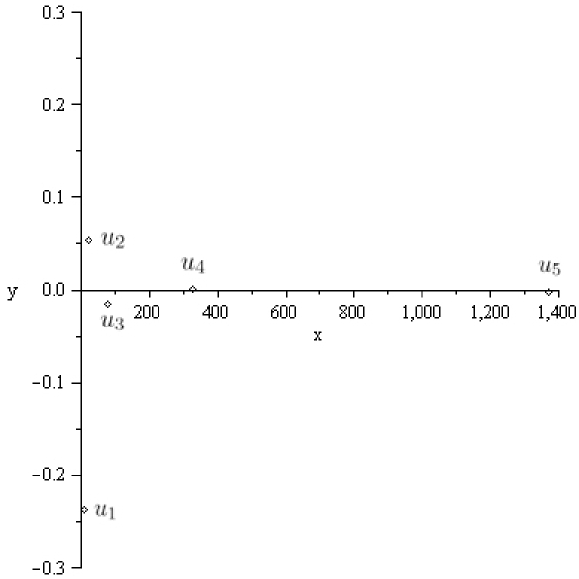

Figure 2.1: Plot point of units $u_1, \ldots, u_5$ under original representation

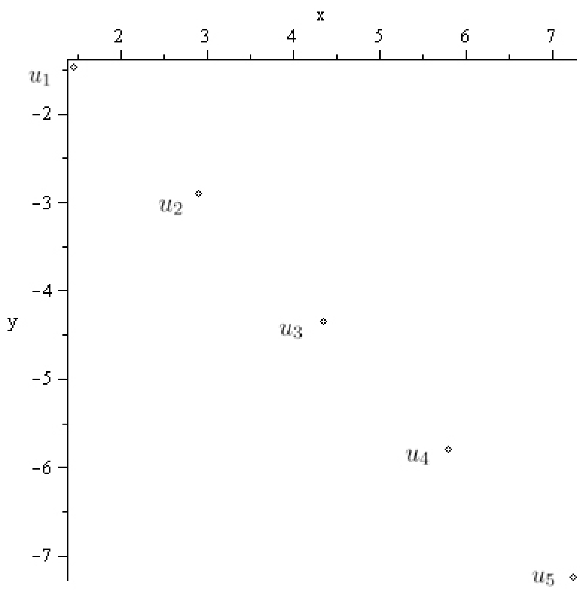

Figure 2.2: Plot point of units $u_1, \ldots, u_5$ under logarithmic representation

2.2.5 The Logarithmic Space

We now explicitly define this new representation using the logarithm and explore some of its properties.

For $x = (x_1, \ldots, x_{r_1+r_2}) \in \Upsilon^{r_1,r_2}$, we define

$$l_i'(x) = \begin{cases} \ln|x_j| & \text{if } j = 1, \ldots, r_1, \\ \ln|x_j|^2 & \text{if } j = r_1+1, \ldots, r_1+r_2. \end{cases}$$

We then associate each point $x \in \Upsilon^{r_1+r_2}$ to the vector

$$l'(x) := (l_1'(x), \ldots, l_{r_1+r_2}'(x)).$$

Which is simply a vector in $\mathbb{R}^{r_1+r_2}$.

So, we can now represent an element $\alpha \in K$ in the logarithmic space by

$$l'(x(\alpha)) = (l_1'(\sigma_1(\alpha)), \ldots, l_{r_1+r_2}'(\sigma_{r_1+r_2}(\alpha))).$$

Notation 2.2.5. *For ease of notation we will denote $l(\alpha) := l'(x(\alpha))$ and denote $l_i(\alpha) := l_i'(\sigma_i(\alpha))$*

Remark 2.2.6. *A full geometric representation of $\alpha \in K$ in the logarithmic space is*

$$l(\alpha) = (ln|(\sigma_1(\alpha))|, \ldots, ln|(\sigma_{r_1}(\alpha))|, ln|(\sigma_{r_1+1}(\alpha))|^2, \ldots, ln|(\sigma_{r_1+r_2}(\alpha))|^2).$$

Proposition 2.2.7. *For $\alpha \in K$, then*

$$\sum_{i=1}^{r_1+r_2} l_i(\alpha) = \ln|N(\alpha)|.$$

Proof. We will first prove a general property of the logarithmic space described and then apply it to the geometric representation of the algebraic number field. Firstly, for $x = (x_1, \ldots, x_{r_1+r_2}) \in \Upsilon^{r_1,r_2}$,

$$\begin{aligned} \sum_{i=1}^{r_1+r_2} l_i'(x) &= \ln|x_1| + \ldots + \ln|x_{r_1}| + \ln|x_{r_1+1}|^2 + \ldots + \ln|x_{r_1+r_2}|^2 \\ &= \ln(|x_1| \cdots |x_{r_1}||x_{r_1+1}|^2 \cdots |x_{r_1+r_2}|^2) \\ &= \ln|(x_1 \cdots x_{r_1}|x_{r_1+1}|^2 \cdots |x_{r_1+r_2}|^2)| \\ &= \ln|N_\Upsilon(x(\alpha))|. \end{aligned}$$

Now, since $x(\alpha) \in \Upsilon^{r_1,r_2}$, and since $N_\Upsilon(x(\alpha)) = N(\alpha)$,

$$\begin{aligned} \sum_{i=1}^{r_1+r_2} l_i(\alpha) &= \sum_{i=1}^{r_1+r_2} l_i'(x(\alpha)) \\ &= \ln|N_\Upsilon(x(\alpha))| \\ &= \ln|N(\alpha)|. \end{aligned}$$

$\square$

Also, this logarithmic representation has some simple properties that enable easy manipulation of representations of elements of K within it. I will state them here without proof as they follow easily from previous properties described. They are, for $\alpha, \beta \in K$, $a \in \mathbb{Q}$,

- $l(\alpha\beta) = l(\alpha) + l(\beta)$

and

- $l(\alpha^a) = al(\alpha)$.

2.2.6 The Representation of Units

Continuing with our representation in the logarithmic space we will now look into the structure of the units within a general algebraic number field.

Taking u to be a general unit in K, of the form $u = \zeta\varepsilon_1^{a_1} \cdots \varepsilon_r^{a_r}$ as described in theorem 2.1.18, we see that since $N(u) = \pm 1$,

$$\sum_{i=1}^{r_1+r_2} l_i(u) = \ln |N(u)| = 0.$$

In other words, for all units, the sum of the coordinates of its representation in the logarithmic space is equal to zero.

Within the ring of integers, $\mathcal{O}_K$ of an algebraic number field, K, there will exist primitive roots of unity, we will denote by A the set of roots of unity in K. Under any of the embeddings, $\sigma_1, \ldots, \sigma_n$, of K into $\mathbb{C}$, the primitive roots are mapped into the unit circle, so

$$|\sigma_i(\alpha)| = 1, \quad \alpha \in A.$$

Thus, under the representation a root of unity, $\alpha \in A$ will be sent to

$$l(\alpha) = (\ln(1), \ldots, \ln(1), \ln|1|^2, \ldots, \ln|1|^2) = (0, \ldots, 0).$$

The roots of unity are **torsion** elements; they have a finite order. Since the roots of unity in the field map to the zero vector, multiplying an element by a root of unity in the logarithmic space does not change the resulting representation. However, $\mathcal{O}_K$ also contains a system of $r = r_1 + r_2 - 1$ fundamental units, $\varepsilon_1, \ldots, \varepsilon_r$, which are not roots of unity and do not have finite order. Since we know that the representation is injective, we obtain a set of r distinct vectors for these fundamental units in the logarithmic space,

$$l(\varepsilon_1), \ldots, l(\varepsilon_r).$$

Next, we examine the representation of a general unit u when expressed in the form outlined in Dirichlet's Theorem.

Proposition 2.2.8. *Let u be a unit in $\mathcal{O}_K$, with $u = \zeta\varepsilon_1^{a_1}\cdots\varepsilon_r^{a_r} \in \mathcal{O}_K$, and $a_1, \ldots, a_r \in \mathbb{Z}$ as in Dirichlet's Theorem. Then*

$$l(u) = a_1 l(\varepsilon_1) + \ldots + a_r l(\varepsilon_r)$$

and is thus a lattice of dimension $r = r_1 + r_2 - 1$.

Proof. We first note that since each embedding is a field homomorphism then for all i,

$$\begin{array}{rcl}
\ln|\sigma_i(u)| & = & \ln|\sigma_i(\zeta\varepsilon_1^{a_1}\cdots\varepsilon_r^{a_r})| \\
& = & \ln|\sigma_i(\zeta)\sigma_i(\varepsilon_1^{a_1})\cdots\sigma_i(\varepsilon_r^{a_r})| \\
& = & \ln(|\sigma_i(\zeta)||\sigma_i(\varepsilon_1^{a_1})|\cdots|\sigma_i(\varepsilon_r^{a_r})|) \\
& = & \ln|\sigma_i(\zeta)| + \ln|\sigma_i(\varepsilon_1^{a_1})| + \ldots + \ln|\sigma_i(\varepsilon_r^{a_r})|) \\
& = & a_1\ln|\sigma_i(\varepsilon_1)| + \ldots + a_r\ln|\sigma_i(\varepsilon_r)|.
\end{array}$$

To make the following more convincing and clear, we have briefly changed notation to display the vectors as column vectors as opposed to the row vectors given before.

$$
\begin{aligned}
l(u) &= l(\zeta\varepsilon_1^{a_1}\cdots\varepsilon_r^{a_r}) \\
&= \begin{pmatrix} \ln|\sigma_1(\zeta\varepsilon_1^{a_1}\cdots\varepsilon_r^{a_r})| \\ \vdots \\ \ln|\sigma_{r_1}(\zeta\varepsilon_1^{a_1}\cdots\varepsilon_r^{a_r})| \\ \ln|\sigma_{r_1+1}(\zeta\varepsilon_1^{a_1}\cdots\varepsilon_r^{a_r})|^2 \\ \vdots \\ \ln|\sigma_{r_1+r_2}(\zeta\varepsilon_1^{a_1}\cdots\varepsilon_r^{a_r})|^2 \end{pmatrix} \\
&= \begin{pmatrix} a_1\ln|\sigma_1(\varepsilon_1)| + \ldots + a_r\ln|\sigma_1(\varepsilon_r)| \\ \vdots \\ a_1\ln|\sigma_{r_1}(\varepsilon_1)| + \ldots + a_r\ln|\sigma_{r_1}(\varepsilon_r)| \\ 2(a_1\ln|\sigma_{r_1+1}(\varepsilon_1)| + \ldots + a_r\ln|\sigma_{r_1+1}(\varepsilon_r)|) \\ \vdots \\ 2(a_1\ln|\sigma_{r_1+r_2}(\varepsilon_1)| + \ldots + a_r\ln|\sigma_{r_1+r_2}(\varepsilon_r)| \end{pmatrix} \\
&= a_1\begin{pmatrix} a_1\ln|\sigma_1(\varepsilon_1)| \\ \vdots \\ \ln|\sigma_{r_1}(\varepsilon_1)| \\ 2(\ln|\sigma_{r_1+1}(\varepsilon_1)|) \\ \vdots \\ 2(\ln|\sigma_{r_1+r_2}(\varepsilon_1)| \end{pmatrix} + \ldots + a_r\begin{pmatrix} \ln|\sigma_1(\varepsilon_r)| \\ \vdots \\ \ln|\sigma_{r_1}(\varepsilon_r)| \\ 2(\ln|\sigma_{r_1+1}(\varepsilon_r)|) \\ \vdots \\ 2(\ln|\sigma_{r_1+r_2}(\varepsilon_r)|) \end{pmatrix} \\
&= a_1 l(\varepsilon_1) + \ldots + a_r l(\varepsilon_r).
\end{aligned}
$$

□

We have now obtained the surprising result that the group of units are the points of a lattice when represented in the logarithmic space.

2.2.7 The Dirichlet Regulator

We are now ready to define the Dirichlet Regulator, R_K, of the algebraic number field K. This is an important invariant of the algebraic number field which gives a sense of the 'size' of the units.

We recall that we obtained vectors representing the units in our system of fundamental units

$$l(\varepsilon_1), \ldots, l(\varepsilon_r),$$

where

$$l(\varepsilon_i) = (l_1(\varepsilon_i), \ldots, l_{r_1+r_2}(\varepsilon_i)).$$

We construct a matrix, A, out of these vectors,

$$A = \begin{pmatrix} l_1(\varepsilon_1) & \dots & l_{r_1+r_2}(\varepsilon_1) \\ \vdots & \ddots & \vdots \\ l_1(\varepsilon_r) & \dots & l_{r_1+r_2}(\varepsilon_r) \end{pmatrix}.$$

Definition 2.2.9. ***The Dirichlet Regulator*** *R_K, of the algebraic number field K is defined to be the determinant of a maximum minor of the matrix A, defined above. Moreover, R_K is independent of the choice of the system of fundamental units and is an invariant of the algebraic number field.*

For the Dirichlet Regulator to be well defined, we must confirm that the determinants of all maximum minors are equal. To show this we use the property that the coordinates of a unit under the logarithmic representation sum to zero. The columns will therefore be linearly dependent and sum to zero. This allows us to change the matrix A with the i-th column removed into the matrix A with the j-th column removed through a series of column operations (with $1 \leq i, j \leq r_1 + r_2$). Since column operations do not affect the determinant of a matrix, we conclude the Dirichlet regulator is well defined.

Relationship to the Volume of a Fundamental Parallelepiped

We can now compare the definition of R_K with the geometric idea of the volume of parallelepipeds. The volume of a parallelepiped is calculated by taking the determinant of the matrix whose rows are made up of the vectors defining the parallelepiped, i.e for a 2-dimensional parallepiped, a parallelogram, the vectors $\mathbf{a} = (a_1, a_2)$ and $\mathbf{b} = (b_1, b_2)$ define the parallelogram shown in figure 2.3.

The volume of this parallelogram is

$$\text{volume} = \det \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}. \tag{2.1}$$

The Dirichlet regulator was obtained in a very similar way. The only difference being that the parallelepipeds defined by the representation of the units lie in a dimension one lower than that of the overall space. We will label one such parallelepiped, P, which will, of course have the same volume as any other parallelepiped in the lattice. The matrix A, above, is the matrix constructed out of the vectors $l(\varepsilon_1), \dots, l(\varepsilon_r)$ in a similar way to (2.1). A has $r_1 + r_2$ columns, but only $r_1 + r_2 - 1$ rows, reflecting the fact that the parallelepiped is of a lower dimension than the space containing it. We create one more vector of length $r_1 + r_2$,

$$v_0 = \frac{1}{\sqrt{r_1 + r_2}}(1, \dots, 1).$$

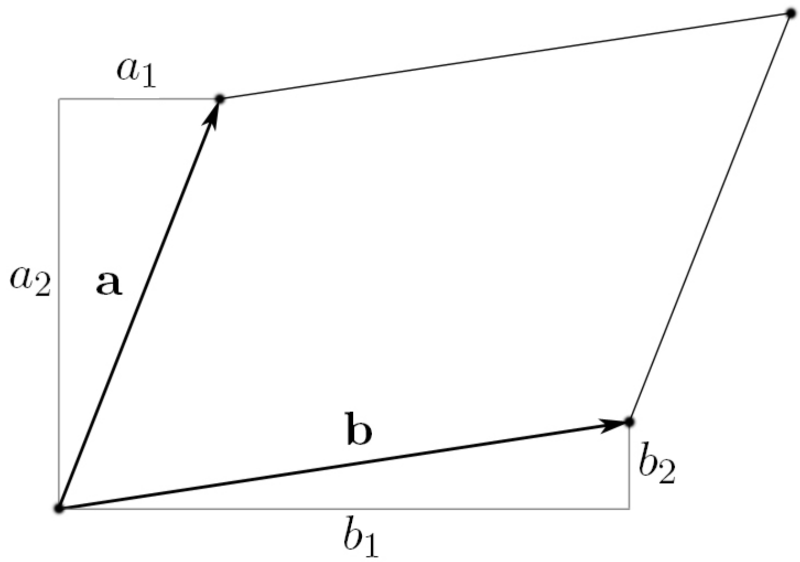

Figure 2.3: Parallelogram defined by **a** and **b**

This is a unit vector and is orthogonal to the parallelepiped since when we dot-product v_0 with $l(\varepsilon_i)$, for any i,

$$\begin{aligned} l(\varepsilon_i).v_0 &= \frac{1}{\sqrt{r_1+r_2}}(l_1(\varepsilon_i),\ldots,l_{r_1+r_2}(\varepsilon_i)).(1,\ldots,1) \\ &= \frac{1}{\sqrt{r_1+r_2}}(l_1(\varepsilon_i)+\ldots+l_{r_1+r_2}(\varepsilon_i)) \end{aligned}$$

and since the coordinates of $l(\varepsilon_i)$ sum to zero, this expression

$$= 0.$$

By making a new parallelepiped, P', out of the vectors $v_0, l(\varepsilon_1), \ldots, l(\varepsilon_r)$ we obtain a (r_1+r_2)-dimensional prism with base P and height 1. We obtain $vol(P') = vol(P) \cdot 1 = vol(P)$. So,

$$vol(P) = \det \begin{pmatrix} \frac{1}{\sqrt{r_1+r_2}} & \cdots & \frac{1}{\sqrt{r_1+r_2}} \\ l_1(\varepsilon_1) & \ldots & l_{r_1+r_2}(\varepsilon_1) \\ \vdots & \ddots & \vdots \\ l_1(\varepsilon_r) & \ldots & l_{r_1+r_2}(\varepsilon_r) \end{pmatrix}.$$

This is simply the sum of the determinant of each maximum minor of A multiplied by $\frac{1}{\sqrt{r_1+r_2}}$. Since we defined the determinant of a maximum minor to infact be the Dirichlet Regulator, R_K, we obtain:

$$vol(P) = (r_1+r_2)\left(\frac{1}{\sqrt{r_1+r_2}}\right) R_K = \sqrt{r_1+r_2} R_K.$$

This interprets the Dirichlet regulator as the volume of a parallelepiped that encompasses information about the units in the algebraic number field.

Chapter 3

A Specific Example

We now apply the procedure of geometrically representing an algebraic number field in a specific example. Seeing this in practice is a very useful way of getting an impression of how it works and the beauty of its simplicity.

3.1 Outline of the Field

The algebraic number field we look at is,

$$K = \mathbb{Q}(\theta) \qquad \text{where} \qquad \theta^7 = 1.$$

This is the cyclotomic field containing the seventh roots of unity.

Notation 3.1.1. *We shall denote the primitive n-th root of unity as ζ_n. Therefore, the primitive seventh root of unity will be denoted ζ_7.*

Proposition 3.1.2. *In our example, $[\mathbb{Q}(\theta) : \mathbb{Q}] = 6$, and the minimal polynomial for θ over $\mathbb{Q}$ is*

$$p_\theta(x) = x^6 + x^5 + x^4 + x^3 + x^2 + x + 1.$$

Proof. θ is the root of the polynomial $x^7 - 1$. To find a minimum polynomial for the extension $\mathbb{Q}(\theta)$ we need to find a polynomial with ζ_7 as a root, and one that is irreducible over $\mathbb{Q}$. $x^7 - 1$ is not irreducible over $\mathbb{Q}$, since $x = 1$ is obviously a root of the polynomial. We therefore factorise to give

$$x^7 - 1 = (x - 1)(x^6 + x^5 + x^4 + x^3 + x^2 + x + 1).$$

Our new candidate for $p_\theta(x)$ is therefore $f(x) = x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$. This polynomial is irreducible due to the rational root theorem (as used in Example 2.1.7), which tells us that the only possible rational roots of this polynomial are ± 1. Since $f(1) = 7$ and $f(-1) = 1$, we conclude that this polynomial is irreducible over $\mathbb{Q}$. $f(x)$ has $x = \zeta_7$ as a root when considered over $\mathbb{C}$ and we conclude that

$$p_\theta(x) = x^6 + x^5 + x^4 + x^3 + x^2 + x + 1.$$

The roots of $p_\theta(x)$ over $\mathbb{C}$ are

$$\zeta_7, \zeta_7^2, \zeta_7^3, \zeta_7^4, \zeta_7^5, \zeta_7^6$$

as shown in figure 3.1. Finally, since the degree of an extension matches the degree of its minimum polynomial then

$$[\mathbb{Q}(\theta) : \mathbb{Q}] = 6.$$

□

Remark 3.1.3. *The minimal polynomial for the n-th primitive root of unity over $\mathbb{Q}$ is the* ***n-th cyclotomic polynomial****, $\varphi_n(x)$, defined to be*

$$\varphi_n(x) = \prod_{\substack{0 \leq a < n \\ gcd(a,n)=1}} (x - \zeta_n^a).$$

With crucial property that

$$x^n - 1 = \prod_{n_1 | n} \varphi_{n_1}(x).$$

This can be used to find the minimum polynomial for any cyclotomic extension. This taken from [2] and [17].

Knowing the degree of the extension and the minimum polynomial is essential to understanding the and manipulating within the field. We now see how the minimum polynomial can be used to simplify the format of a general element in K.

Remark 3.1.4. *We have that*

$$\mathbb{Q}(\theta) = \{a_0 + a_1\theta + a_2\theta^2 + a_3\theta^3 + a_4\theta^4 + a_5\theta^5 + a_6\theta^6 \mid a_i \in \mathbb{Q}, \forall i\}.$$

However, since θ is a root of $p_\theta(x)$, then we have the equation

$$\theta^6 + \theta^5 + \theta^4 + \theta^3 + \theta^2 + \theta + 1 = 0 \quad \Rightarrow \quad \theta^6 = -\theta^5 - \theta^4 - \theta^3 - \theta^2 - \theta - 1.$$

This implies that we had a superfluous term, so we can express $\mathbb{Q}(\theta)$ in the form

$$\mathbb{Q}(\theta) = \{b_0 + b_1\theta + b_2\theta^2 + b_3\theta^3 + b_4\theta^4 + b_5\theta^5 \mid b_i \in \mathbb{Q}, \forall i\}.$$

The algebraic integers can be expressed in the form

$$\{a_0 + a_1\theta + a_2\theta^2 + a_3\theta^3 + a_4\theta^4 + a_5\theta^5 \mid a_i \in \mathbb{Z}, \forall i\}.$$

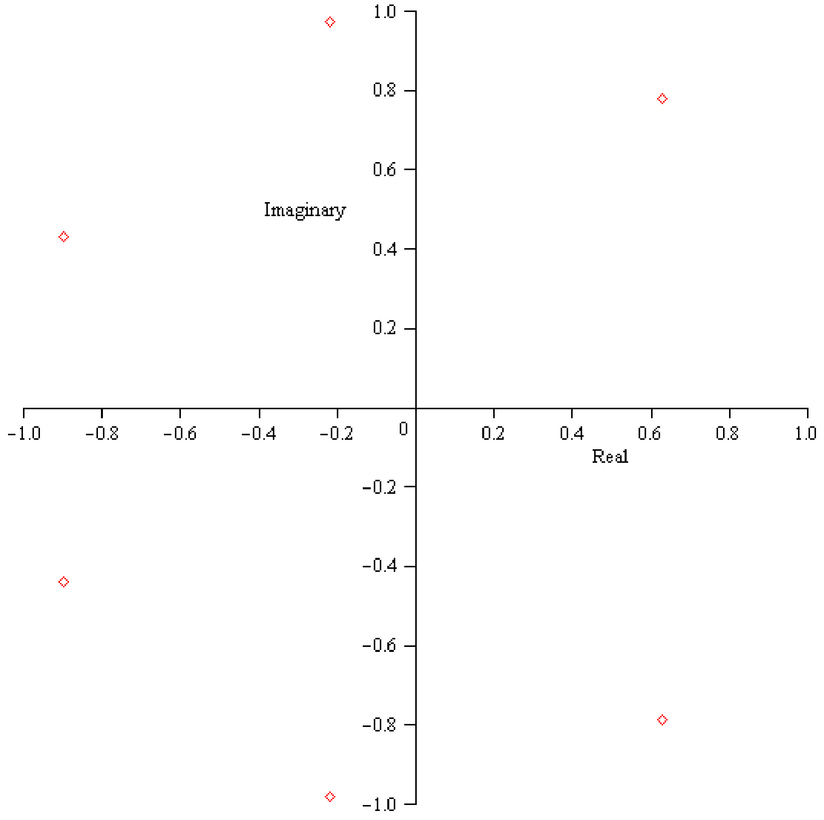

Figure 3.1: A plot of $\zeta_7, \zeta_7^2, \zeta_7^3, \zeta_7^4, \zeta_7^5, \zeta_7^6$

Next, it is easy to see that $r_1 = 0$ and $r_3 = 3$ from figure 3.1, giving us six embeddings from $\mathbb{Q}(\theta)$ into $\mathbb{C}$

$$\begin{aligned}
\sigma_1 &: \mathbb{Q}(\theta) \longmapsto \mathbb{Q}(\zeta_7), \\
\sigma_2 &: \mathbb{Q}(\theta) \longmapsto \mathbb{Q}(\zeta_7^2), \\
\sigma_3 &: \mathbb{Q}(\theta) \longmapsto \mathbb{Q}(\zeta_7^3), \\
\sigma_4 = \overline{\sigma_3} &: \mathbb{Q}(\theta) \longmapsto \mathbb{Q}(\zeta_7^4), \\
\sigma_5 = \overline{\sigma_2} &: \mathbb{Q}(\theta) \longmapsto \mathbb{Q}(\zeta_7^5), \\
\sigma_6 = \overline{\sigma_1} &: \mathbb{Q}(\theta) \longmapsto \mathbb{Q}(\zeta_7^6).
\end{aligned}$$

However, from these description of the embeddings it is not entirely clear how they actually act on an element of $\mathbb{Q}(\theta)$. A better way to view them is via their action on θ. We will examine only the first three since the latter three are merely conjugates of the first and will later be ignored. We have

$$\begin{aligned}
\sigma_1(\theta) &= \zeta_7, \\
\sigma_2(\theta) &= \zeta_7^2, \\
\sigma_3(\theta) &= \zeta_7^3.
\end{aligned}$$

We also note that these complex embeddings have no effect on real numbers, so

$$\sigma_i(a) = a, \quad \forall a \in \mathbb{R},\ i = 1, \dots, 6.$$

We then have, for $c_i \in \mathbb{Q}\ \forall i$

$$\begin{aligned}
\sigma_1(c_0 + c_1\theta + c_2\theta^2 + c_3\theta^3 + c_4\theta^4 + c_5\theta^5) &= c_0 + c_1\zeta_7 + c_2\zeta_7^2 + c_3\zeta_7^3 + c_4\zeta_7^4 + c_5\zeta_7^5, \\
\sigma_2(c_0 + c_1\theta + c_2\theta^2 + c_3\theta^3 + c_4\theta^4 + c_5\theta^5) &= c_0 + c_1\zeta_7^2 + c_2\zeta_7^4 + c_3\zeta_7^6 + c_4\zeta_7 + c_5\zeta_7^3 \\
&= c_0 + c_4\zeta_7 + c_1\zeta_7^2 + +c_5\zeta_7^3 + c_2\zeta_7^4 \\
&\quad + c_3(-\zeta_7^5 - \zeta_7^4 - \zeta_7^3 - \zeta_7^2 - \zeta_7 - 1) \\
&= c_0 - c_3 + (c_4 - c_3)\zeta_7 + (c_1 - c_3)\zeta_7^2 \\
&\quad + (c_5 - c_3)\zeta_7^3 + (c_2 - c_3)\zeta_7^4 - c_3\zeta_7^5
\end{aligned}$$

and

$$\begin{aligned}
\sigma_3(c_0 + c_1\theta + c_2\theta^2 + c_3\theta^3 + c_4\theta^4 + c_5\theta^5) &= c_0 + c_5\zeta_7 + c_3\zeta_7^2 + c_1\zeta_7^3 + c_4\zeta_7^5 \\
&\quad + c_2(-\zeta_7^5 - \zeta_7^4 - \zeta_7^3 - \zeta_7^2 - \zeta_7 - 1) \\
&= c_0 - c_2 + (c_5 - c_2)\zeta_7 + (c_3 - c_2)\zeta_7^2 \\
&\quad + (c_1 - c_2)\zeta_7^3 - c_2\zeta_7^4 + (c_4 - c_2)\zeta_7^5.
\end{aligned}$$

Now that we have a good understanding of some of the basic properties of this field and how we can embed it into the complex numbers, we geometrically represent it as outlined in chapter 2.

3.2 Geometric Representation of $\mathbb{Q}(\theta)$ in the Space $\Upsilon^{0,3}$ and the Logarithmic Space

Since $r_1 = 0$ the space $\Upsilon^{0,3}$ is in fact exactly the space $\mathbb{C}^3$ with the usual basis. We now give the geometric representation of the element $\alpha \in \mathbb{Q}(\theta)$ in the space $\Upsilon^{0,3}$ by

$$x(\alpha) = (\sigma_1(\alpha), \sigma_2(\alpha), \sigma_3(\alpha)).$$

As before, we express the norm of an element $x(\alpha) \in \Upsilon^{0,3}$ by $N_\Upsilon(x(\alpha))$. So, for $x(\alpha) = (x_1, x_2, x_3)$ we have

$$N_\Upsilon(x(\alpha)) = |x_1|^2|x_2|^2|x_3|^2.$$

We also defined, in section 2.1.2, the norm of an element in an algebraic number field. As established, the algebraic number field has a basis $(1, \theta, \theta^2, \theta^3, \theta^4, \theta^5)$, with general element $a + b\theta + c\theta^2 + d\theta^3 + e\theta^4 + f\theta^5$ with $a, b, c, d, e, f \in \mathbb{Q}$. We then use the method described previously to generate a matrix. Therefore, we obtain the following expression for the norm of a general element in $\mathbb{Q}(\theta)$

$$N(a + b\theta + c\theta^2 + d\theta^3 + e\theta^4 + f\theta^5) = det \begin{pmatrix} a & -f & f-e & e-d & d-c & c-b \\ b & a-f & -e & f-d & e-c & d-b \\ c & b-f & a-e & -d & f-c & e-b \\ d & c-f & b-e & a-d & -c & f-b \\ e & d-f & c-e & b-d & a-c & -b \\ f & e-f & d-e & c-d & b-c & a-b \end{pmatrix}.$$

We do not give the full expression of this formula as it is fairly large; the above description shall suffice.

We can now easily check that the two valuations of the norm of an element give the same value, as demonstrated in the following example.

Example 3.2.1. *Let $\mu = 1 + 2\theta + 5\theta^2$. We will now show that $N_\Upsilon(x(\mu)) = N(\mu)$. Firstly,*

$$\begin{aligned} N_\Upsilon(x(\mu)) &= N_\Upsilon((\sigma_1(\mu), \sigma_2(\mu), \sigma_3(\mu))) \\ &= N_\Upsilon((1 + 2\zeta_7 + 5\zeta_7^2, 1 + 2\zeta_7^2 + 5\zeta_7^4, 1 + 2\zeta_7^3 + 5\zeta_7^6)) \\ &= |1 + 2\zeta_7 + 5\zeta_7^2|^2 \cdot |1 + 2\zeta_7^2 + 5\zeta_7^4|^2 \cdot |1 + 2\zeta_7^3 + 5\zeta_7^6|^2 \\ &= 9773. \end{aligned}$$

Conversely, we calculate the norm using the method described above with $a = 1, b = 2, c =$

$5, d = 0, e = 0, f = 0$.

$$
\begin{aligned}
N(\mu) &= det \begin{pmatrix} 1 & 0 & 0 & 0 & -5 & 3 \\ 2 & 1 & 0 & 0 & -5 & -2 \\ 5 & 2 & 1 & 0 & -5 & -2 \\ 0 & 5 & 2 & 1 & -5 & -2 \\ 0 & 0 & 5 & 2 & -4 & -2 \\ 0 & 0 & 0 & 5 & -3 & -1 \end{pmatrix} \\
&= 9773 \\
&= N_{\Upsilon}(x(\mu)).
\end{aligned}
$$

So, for μ, our evaluations of the norm agree. The above determinant was calculated using MAPLE [1], the code can be found in Appendix section B.1.

We now move the representation into the logarithmic space. Since $r_1 = 0$ then for $\alpha \in \mathbb{Q}(\theta)$, we have $l_i(\alpha) = \ln|\sigma_i(\alpha)|^2$ for all i. Our full representation of α in the logarithmic space is

$$l(\alpha) = (l_1(\alpha), l_2(\alpha), l_3(\alpha)) = (\ln|\sigma_1(\alpha)|^2, \ln|\sigma_2(\alpha)|^2, \ln|\sigma_3(\alpha)|^2).$$

3.3 Representation of Units in the Field

In our quest to obtain a full idea of the structure of this representation we look at the units, and more specifically the fundamental units of our specific algebraic number field.

3.3.1 Units of a Cyclotomic Field

Fourteen of the units in our field are fairly obvious, namely

$$\pm 1, \pm\zeta_7, \pm\zeta_7^2, \pm\zeta_7^3, \pm\zeta_7^4, \pm\zeta_7^5, \pm\zeta_7^6.$$

As they are all roots of unity, these units are mapped under the representation in the logarithmic space to the zero vector. These elements are torsion, since they have finite order. Dirichlet's Theorem shows us the existence of two more units (since $r = r_1 + r_2 - 1 = 3 - 1 = 2$), which are non-torsion. Modified from [19], page 144, we state without proof the following Lemma:

Lemma 3.3.1. *Let p be a prime, $m \geq 1$ and θ be a p^m-th root of unity. The cyclotomic units of $\mathbb{Q}(\theta)$ are generated by -1, ζ_{p^m} and the units*

$$\xi_a = \theta^{\left(\frac{1-a}{2}\right)} \frac{1-\theta^a}{1-\theta} \qquad 1 < a < \frac{p^m}{2},\ (a, p) = 1.$$

For the seventh cyclotomic field we take $p = 7$, $m = 1$. We have already noted that the torsion units are generated by -1 and ζ_7. From the Lemma, the possible a values for $\mathbb{Q}(\theta)$ are $a = 2, 3$, so our other two required units will be ξ_2 and ξ_3. Since we know that multiplying an element of an algebraic number field by roots of unity does not change their representation in the logarithmic space, we will disregard the first factor in the expression of ξ_2 and ξ_3 and arrive at our required, non-torsion, independent, units

$$u_1 = \frac{1-\theta^2}{1-\theta} = 1+\theta \quad \text{and} \quad u_2 = \frac{1-\theta^3}{1-\theta} = 1+\theta+\theta^2.$$

We now know all required information about the seventh cyclotomic field to numerically calculate the representation of the units and see the resulting lattice.

3.3.2 Numerical Evaluation

We now use the computer package MAPLE [1] to calculate the representation of the units. We only give the necessary results here, the reader is referred to Appendix section B.1 for the full program code. Also, note that these are only approximate numerical evaluations, which is why the symbol '$\approx$' is used throughout.

We see that, under the logarithmic representation, the above non-torsion units u_1 and u_2 give the vectors

$$l(u_1) \approx (1.177725212, 0.4414486205, -1.619173834) \in \mathbb{R}^3$$

and

$$l(u_2) \approx (1.619173832, -1.177725211, -0.4414486200) \in \mathbb{R}^3.$$

Note that, as expected, the coordinates sum to zero in both u_1 and u_2

$$1.177725212 + 0.4414486205 - 1.619173834 = 1.619173832 - 1.177725211 - 0.4414486200 \approx 0.$$

The points $l(u_1)$ and $l(u_2)$ are plotted in figure 3.2. We can then generate the lattice, with lattice points representing the units of K. Figure 3.3 shows 25 points of this lattice. The image has been slightly modified from the Maple output to include lattice lines.

We then form a matrix from the representations of our fundamental units u_1 and u_2, giving

$$\begin{pmatrix} 1.177725212 & 0.4414486205 & -1.619173834 \\ 1.619173832 & -1.177725211 & -0.4414486200 \end{pmatrix}.$$

The determinant of each maximum minor of this matrix is the same (confirmed numerically in appendix section B.1). Taking one of the minors we obtain

$$R_K \approx \begin{pmatrix} 0.4414486205 & -1.619173834 \\ -1.177725211 & -0.4414486200 \end{pmatrix} \approx 2.101818729.$$

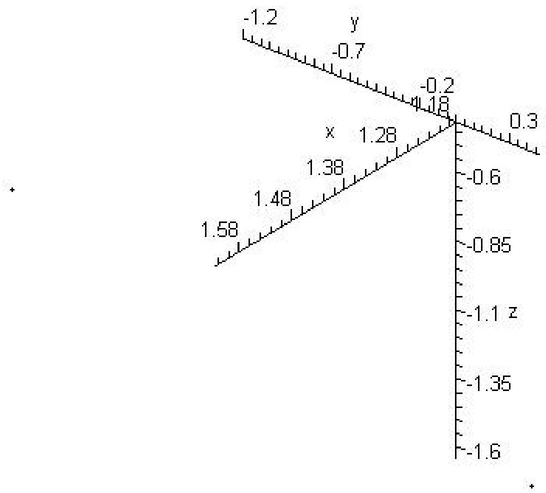

Figure 3.2: Point plot of $l(u_1)$ and $l(u_2)$

We have successfully found an evaluation of the Dirichlet regulator for K. We can also evaluate the volume (in this case it is an area) of a fundamental parallelogram, P of the lattice,

$$vol(P) = R_K\sqrt{3} \approx 3.640456828$$

where the $\sqrt{3}$ is the value $\sqrt{r_1 + r_2}$ (see section 2.2.7).

We have now given an example of constructing a geometric representation and calculating the Dirichlet regulator for a specific algebraic number field. The importance of the Dirichlet regulator becomes evident when we see the result of the following chapter, the analytic class number formula.

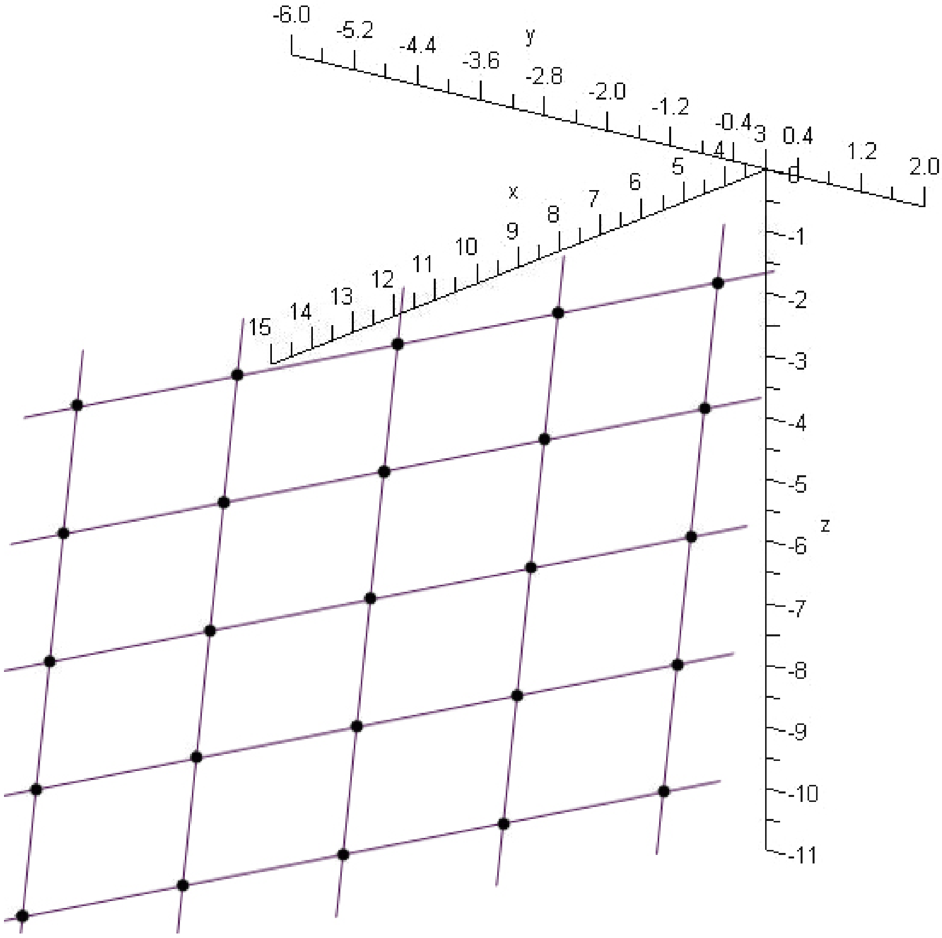

Figure 3.3: Lattice with lattice points representing the units of the seventh cyclotomic field

Chapter 4

The Analytic Class Number Formula

4.1 The Dedekind Zeta Function

We now introduce the Dedekind zeta function, named after the German Mathematician Julius Wilhelm Richard Dedekind (1831 - 1916). It can be viewed as a generalisation of the more well known Riemann zeta function. The Riemann zeta function provides a useful introduction to the function, so it is here that we will start.

4.1.1 The Riemann Zeta Function

Definition 4.1.1. *The* ***Riemann zeta function****, $\zeta(s)$, is defined by the infinite series*

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \qquad s \in \mathbb{C}.$$

The Riemann zeta function is well known through popularisation of the Riemann hypothesis, one of the seven Millenium Prize Problems [1]. The Riemann hypothesis concerns the location of the non-trivial zeros of Riemann zeta function, the hypothesis states that they all lie on the vertical line where the real part equals $\frac{1}{2}$.

Remark 4.1.2. *The Riemann zeta function can be expressed in terms of polylogarithms since*

$$\zeta(s) = Li_s(1).$$

[1]Established by the Clay Mathematics Institute, a solution to each one of the seven problems has a \$1,000,000 prize attached.

4.1.2 The Definition of the Dedekind Zeta Function

Definition 4.1.3. *We define the **Dedekind Zeta function** of an algebraic number field K to be*

$$\zeta_K(s) = \sum_{\mathfrak{a}} \frac{1}{N(\mathfrak{a})^s} \quad s \in \mathbb{C}.$$

The summation runs over all $\mathfrak{a}$, which are ideals in $\mathcal{O}_K$ and $N(\mathfrak{a})$ is the ideal norm of $\mathfrak{a}$.

We note that the Dedekind zeta function is a natural generalisation of the Riemann zeta function, where instead of $\mathbb{C}$ we run the summation over the ring of integers, $\mathcal{O}_K$, and use the ideal norm on the denominator of the fraction in the summand. We also note that the Dedekind zeta function for the field $K = \mathbb{Q}$ exactly coincides with the Riemann zeta function.

The series defining the Dedekind zeta function converges for values of s with $\mathrm{Re}(s) > 1$, but it is possible to extend analytically the Dedekind zeta function to the entire complex plane except for its pole at $s = 1$. This pole is a simple pole, and its residue is central to the analytic class number formula.

4.1.3 Important Properties of the Dedekind Zeta Function

For the purposes of the analytic class number formula we only need to evaluate the Dedekind zeta function, $\zeta_K(s)$ at values $s = 0$ and $s = 1$. However it is useful to know some of the properties of the Dedekind zeta function as they are used later in this report.

Relationship between $\zeta_K(s)$ and $\zeta_K(1-s)$

This relationship allows us to evaluate, in a sense, the Dedekind zeta function at negative values. This seems strange at first since the series will obviously diverge, but we will see that it is possible to calculate an idea of a value via a reflection property.

We first examine the specific example of a Dedekind zeta function for $K = \mathbb{Q}$, the Riemann zeta function, $\zeta(s)$. From chapter 3 of [5] we find that by modifying the Riemann zeta function very slightly we obtain a symmetric function, denoted $\xi(s)$, defined by

$$\xi(s) = \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s).$$

In other words, the function $\xi(s)$ satisfies

$$\xi(s) = \xi(1-s).$$

Remark 4.1.4. *The function $\Gamma(z)$ is the Gamma function defined*

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt \quad n \in \{\mathbb{C} - \mathbb{Z}_{\leq 0}\}.$$

The Gamma function has the important property that for $n \in \mathbb{Z}_{>0}$

$$\Gamma(n) = (n-1)!.$$

By comparing the definition of $\xi(s)$ for s and $1-s$ we obtain a reflection functional equation for the Riemann zeta function

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\zeta(1-s).$$

A similar property holds for the Dedekind zeta function and we are able to define a generalisation of the function $\xi(s)$, which we will denote $\xi_K(s)$. The function $\xi_K(s)$ is also symmetric.

Proposition 4.1.5. *The function $\xi_K(s)$, a modification of the Dedekind zeta functions is a symmetric function. $\xi_K(s)$ is defined, for an algebraic number field K with discriminant d_K by*

$$\xi_K(s) = \left(\frac{|d_K|}{2^{2r_2}\pi^{r_1+2r_2}}\right)^{\frac{s}{2}} \Gamma\left(\frac{s}{2}\right)^{r_1} \Gamma(s)^{r_2} \zeta_K(s).$$

We will not prove this here, instead we invite the reader to see [15] and [16] for more details.

$\zeta_K(s)$ at $s = 0$ and $s = 1$

As previously mentioned, the Dedekind zeta function has a simple pole at $s = 1$ and we can use the above functional equation developed to also find a value at s=0.

Proposition 4.1.6. *The Dedekind zeta function has a zero at $s = 0$ of order $r_1 + r_2 - 1$.*

Proof. For ease of notation we let $\xi_K(s) = f(s) \cdot \zeta_K(s)$ where

$$f(s) = \left(\frac{|d_K|}{2^{2r_2}\pi^{r_1+2r_2}}\right)^{\frac{s}{2}} \Gamma\left(\frac{s}{2}\right)^{r_1} \Gamma(s)^{r_2}.$$

Evaluating the function $f(s)$ at $s = 1$

$$f(1) = \left(\frac{|d_K|}{2^{2r_2}\pi^{r_1+2r_2}}\right)^{\frac{1}{2}} \Gamma\left(\frac{1}{2}\right)^{r_1} \Gamma(1)^{r_2}.$$

Using the fact that $\Gamma(1) = 0! = 1$ and the well known result that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$, we obtain

$$f(1) = \left(\frac{|d_K|}{2^{2r_2}\pi^{r_1+2r_2}}\right)^{\frac{1}{2}} (\sqrt{\pi})^{r_1}.$$

Since d_K is non-zero, we conclude that $f(s)$ has neither a pole nor zero at $s = 1$, meaning that since $\zeta_K(s)$ has a simple pole at $s = 1$, then $\xi_K(s)$ does. We also find that since the Gamma function has a simple pole at $s = 0$, then $f(s)$ has a pole of order $r_1 + r_2$ at $s = 0$. Using the functional equation for $\xi_K(s)$ we obtain, in a sense,

$$\zeta_K(0) = \frac{f(1)}{f(0)}\zeta_K(1).$$

The simple pole of $\zeta_K(s)$ at $s = 1$ in a sense 'cancels' with one of the poles of $f(s)$ at $s = 0$, giving us the required result that $\zeta_K(s)$ has a zero of order $r_1 + r_2 - 1$ at $s = 0$. □

4.2 The Analytic Class Number Formula

Theorem 4.2.1. *The* ***analytic class number formula*** *is*

$$res_{s=1}\zeta_K(s) = \frac{2^{r_1+r_2}\pi^{r_2}h_K R_K}{w_K\sqrt{|d_K|}}$$

where

- h_K *is the class number of* K*.*
- w_K *is the number of roots of unity contained in* K*.*
- R_K *is the Dirichlet Regulator from chapter 2.*
- d_K *is the discriminant of* K*.*

The formula encompasses several of the most important invariants of an algebraic number field and must surely be one of the most beautiful results in algebraic number theory.

However, the analytic class number formula can be expressed in an even simpler way using the reflection functional equation of the Dedekind zeta function. If we instead examine the residue of the zero at $s = 0$ then using the result for the residue of the pole at $s = 1$ we find that

$$\lim_{s\to 0}\frac{\zeta_K(s)}{s^{r_1+r_2-1}} = -\frac{h_K R_K}{w_K}.$$

This alternative version of the analytic class number formula is easier to compute and we will numerically verify this version in the following section.

Notation 4.2.2. *For ease of notation we denote*

$$\zeta_K^*(0) = \lim_{s\to 0}\frac{\zeta_K(s)}{s^{r_1+r_2-1}}.$$

So the analytic class number formula reads

$$\zeta_K^*(0) = -\frac{h_K R_K}{w_K}.$$

4.3 Verifying the Formula

In the previous chapter we introduced a specific example which we explored in detail, and evaluated the Dirichlet regulator. The example we used was the cyclotomic field for the seventh roots of unity, $K = \mathbb{Q}(\theta)$ where θ is a root of the polynomial $p_\theta(x) = x^6+x^5+x^4+x^3+x^2+x+1$. We recall that we made a numerical evaluation of the Dirichlet regulator for this algebraic number field,

$$R_K \approx 2.101818729.$$

We can now use this value to make a numerical confirmation of the analytic class number formula.

The other two invariants required on the right hand side of the formula are the class number and the number of roots of unity contained within K. The set of roots of unity in K are generated by -1 and ζ_7, where ζ_7 is the primitive seventh root of unity, giving us $w_K = 14$. We also use the fact that the class number is $h_K = 1$ for the seventh cyclotomic field. We will not describe here the reason for this, we briefly discussed class numbers in section 2.1.2 and a table of class numbers for cyclotomic fields can be found on page 429 of [4].

We then evaluate

$$\lim_{s\to 0} \frac{\zeta_K(s)}{s^{(r_1+r_2-1)}} = -\frac{1 \cdot 2.101818729}{14} = -0.1501299092.$$

We can now use the software GP/PARI [8] to evaluate the value of the residue of the zero of the Dedekind zeta function for K at 0. Appendix B.2 shows an approximate numerical evaluation for this value is

$$\lim_{s\to 0} \frac{\zeta_K(s)}{s^{(r_1+r_2-1)}} = -0.1501299092.$$

These evaluations give a numerical confirmation of the analytic class number formula for this algebraic number field.

4.4 Motivation to go Higher

The result of the Analytic class number formula gives us an evaluation of the Dedekind zeta function, $\zeta_K(s)$, at the value $s = 1$, or, as we have seen, at $s = 0$. We observe that we used the natural logarithm in the calculation of the regulator of K. We recall that the logarithm is similar to $Li_1(z)$

$$\mathrm{Li}_1(z) = -\ln(1-z).$$

The natural thought of a curious mathemtician is whether the analytic class number formula is merely a specific case of a grander idea?

We will think of the analytic class number formula as the case $m = 1$ due to its relation to $Li_m(z)$ when $m = 1$ and Dedekind zeta function $\zeta_K(s)$ at $s = 1$. The terms in the formula were

- The Dedekind zeta value, $\zeta_K^*(0)$.
- The class number h_K of K.
- The number of roots of unity w_K in K.
- The Dirichlet Regulator R_K for K.

We attempt to find 'higher' analogues for each of these terms. The easiest to raise to a higher case is the Dedekind zeta value.

4.4.1 'Higher' Dedekind zeta Values

'Higher' Dedekind zeta values are actually easier to interpret as $\zeta_K(s)$ is neither a pole, nor a zero for integers $s > 1$. We will calculate these using GP/PARI [8] as seen in Appendix section B.3.3.

4.4.2 'Higher' Class Numbers and 'Higher' Roots of Unity

These values are obtained from the relatively new algebraic K-theory. We do not discuss this topic in this report, however the inspired reader is pointed in the direction of [14] and [11]. In short, algebraic K-theory can be used to attach a sequence of abelian groups (groups with a commutative group operation), called the algebraic K-groups, to the number ring associated with an algebraic number field. From this 'higher' class numbers, and groups of 'higher' roots of unity can be found. Both the 'higher' class number and the number of 'higher' roots of unity will be finite integers and so will materialise as a rational multiplicative difference between the 'higher' regulator and the 'higher' Dedekind zeta value.

4.4.3 'Higher' Regulators

The Dirichlet Regulator was obtained from our geometric representation of an algebraic number field in a 'logarithmic' space. This motivates us to ask if we can use higher logarithms to obtain higher regulators. Indeed we can. We first concentrate on the case $m = 2$ and so naturally expect to use the dilogarithm. We will actually use a modification of the dilogarithm, the Bloch-Wigner function. The Bloch-Wigner function is then applied to 'higher units' which are in fact elements of the Bloch group. We discuss the Bloch-Wigner function, the Bloch group and regulators in the case $m = 2$ in the following chapter.

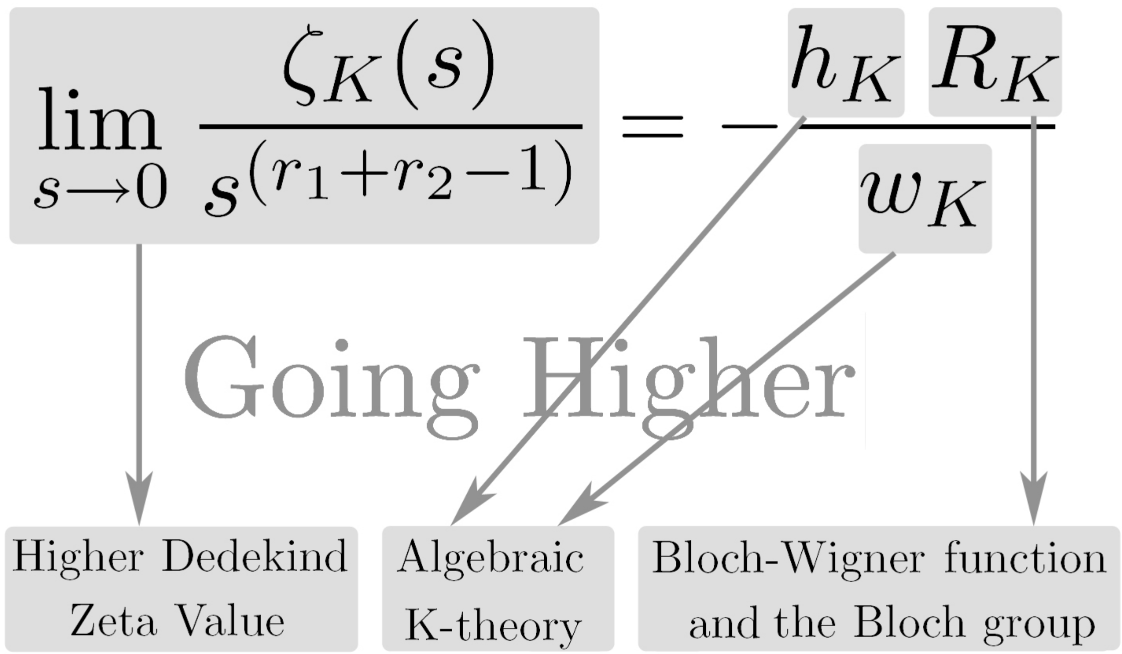

Figure 4.1: Going higher

4.4.4 Summary of Motivation

To clarify the motivating idea, figure 4.1 shows how we consider the move from case $m = 1$, the analytic class number formula, to the case $m = 2$.

Chapter 5

Regulators via the Bloch-Wigner Function and Zagier's Conjecture

In this chapter we introduce the Bloch-Wigner function, a modification of the dilogarithm, and we see how it can be used to formulate 'higher' regulators, which we then use to find a 'higher' analogue of the analytic class number formula.

5.1 The Bloch-Wigner Function

We begin by defining a modification of the dilogarithm, the Bloch-Wigner Function.

Definition 5.1.1. The Bloch-Wigner function*, $D(z)$ is defined to be:*

$$D(z) = \Im(Li_2(z)) + \arg(1-z)\ln|z|.$$

This arises when we consider the imaginary part of the dilogarithm, $Li_2(z)$, extended onto $\mathbb{C} - (1, \infty)$ as in Chapter 1, and modified to a continuous function. As stated in [20], when the function crosses $(1, \infty)$ it jumps by $2\pi i \ln|z|$ giving rise to the continuous function

$$Li_2(z) + i\arg(1-z)\ln|z|.$$

The Bloch-Wigner function has some very important properties and advantages over the dilogarithm. We invite the reader to read Appendix section A.1, where we cover some of the functional equations of the dilogarithm and how replacing the dilogarithm with the Bloch-Wigner function simplifies them.

5.2 The Bloch Group

When we calculated the Dirichlet regulator, we took logarithms of the representations of units of the algebraic number field. Elements of the Bloch group serve as 'higher' units. We will later apply the Bloch-Wigner function to these on our way to finding the 'higher' regulators. The definition of the Bloch group will come later, however elements in the Bloch group will be elements in the group $\mathbb{Z}[K^*]$. We define this now.

Definition 5.2.1. *The group $\mathbb{Z}[K^*]$ is defined as*

$$\mathbb{Z}[K^*] = \left\{ \sum_i n_i[x_i] \mid n_i \in \mathbb{Z},\ x_i \in K^* \right\}.$$

Elements in $\mathbb{Z}[K^*]$ are formal linear combinations of elements in the set K^*.

We next a define an important map, $\beta_2 : \mathbb{Z}[K^*] \longrightarrow \bigwedge^2 K^*$. The space $\bigwedge^2 K^*$ denotes the space of elements of the form $x \wedge y$ with $x, y \in K^* = K - 0$. The symbol '$\wedge$' denotes the wedge product, which is constructed in Appendix section A.2. If new to them, the reader should familiarise themselves with manipulating the wedge product before moving on. Properties of the wedge product (such as $1 \wedge a = 0$, $\forall a \in K$) are assumed in the following sections.

Definition 5.2.2. *We define the map β_2 to be*

$$\begin{aligned} \beta_2 : \mathbb{Z}[K^*] &\longrightarrow \bigwedge^2 K^* \\ [x] &\longmapsto \begin{cases} x \wedge (1-x) & \text{if } x \neq 1 \\ 0 & \text{if } x = 1. \end{cases} \end{aligned}$$

Of particular interest are the elements in the kernel of this map. In Appendix section A.2 are some example of elements in this kernel for the algebraic number field $K = \mathbb{Q}$. In fact, these elements are in the kernel of the map β_2 for every algebraic number field since $\forall K$, $\mathbb{Q} \subset K$.

There are certain elements of $\mathbb{Z}[K^*]$ which are in $\ker \beta_2$ for every algebraic number field, K. We will refer to this kind of element as 'universal elements'.

Proposition 5.2.3. *For any $x \in K$ then*

$$[x^2] - 2[x] - 2[-x] \in \ker \beta_2$$

and more generally, for all $N \in \mathbb{N}$

$$[x^N] - N \sum_{\zeta^N = 1} [\zeta x] \in \ker \beta_2.$$

Proof. The first is easy to show. Under the map

$$\begin{aligned}
[x^2] - 2[x] - 2[-x] &\longmapsto x^2 \wedge (1-x^2) - 2(x \wedge (1-x)) - 2(-x \wedge (1+x)) \\
&= (x^2 \wedge (1-x)(1+x)) - (x^2 \wedge (1-x)) - 2(x^2 \wedge (1+x)) \\
&= (x^2 \wedge (1-x)) + (x^2 \wedge (1+x)) - (x^2 \wedge (1-x)) \\
&\quad -2(x^2 \wedge (1+x)) \\
&= 0.
\end{aligned}$$

The more general formula, the above merely being case $N = 2$, can be proved by noting that

$$1 - x^N = (1-x)(1-\zeta_N x)(1-\zeta_N^2 x)\cdots(1-\zeta_N^{N-1}x)$$

where ζ_N denotes the primitive N-th root of unity. It then follows that

$$\begin{aligned}
[x^N] - N\sum_{\zeta^N=1}[\zeta x] &\longmapsto x^N \wedge (1-x^N) - N\sum_{\zeta^N=1}(\zeta x \wedge (1-\zeta x)) \\
&= N(x \wedge (1-x)(1-\zeta_N x)(1-\zeta_N^2 x)\cdots(1-\zeta_N^{N-1}x)) \\
&\quad -N\sum_{\zeta^N=1}(\zeta x \wedge (1-\zeta x)) \\
&= N\sum_{\zeta^N=1}(\zeta x \wedge (1-\zeta x)) - N\sum_{\zeta^N=1}(\zeta x \wedge (1-\zeta x)) \\
&= 0.
\end{aligned}$$

□

We also note that these expressions are very similar to existing functional equations for the Bloch-Wigner Function, namely the distribution functions. This takes the form

$$D(x^N) - N\sum_{\zeta^N=1} D(\zeta x) \in \ker \beta_2.$$

Other elements of the Bloch group can also be formed using functional equations of the Bloch-Wigner-function. We recall the five-term relation of the Bloch-Wigner function

$$D(x) + D(y) + D\left(\frac{1-x}{1-xy}\right) + D(1-xy) + D\left(\frac{1-y}{1-xy}\right) = 0.$$

We ask, can this also be made into an element of the Bloch group? Incredibly, it can.

Proposition 5.2.4. *Let*

$$\rho = [x] + [y] + \left[\frac{1-x}{1-xy}\right] + [1-xy] + \left[\frac{1-y}{1-xy}\right] \in \mathbb{Z}[K^*]$$

then $\rho \in \ker \beta_2$.

Proof. Under the map β_2

$$\begin{aligned}
\rho \longmapsto\ & x \wedge (1-x) + y \wedge (1-y) + \left(\frac{1-x}{1-xy}\right) \wedge \left(\frac{x-xy}{1-xy}\right) + (1-xy) \wedge xy \\
& + \left(\frac{1-y}{1-xy}\right) \wedge \left(\frac{y-xy}{1-xy}\right) \\
=\ & x \wedge (1-x) + y \wedge (1-y) + (1-x) \wedge (x-xy) - (1-xy) \wedge (x-xy) \\
& - (1-x) \wedge (1-xy) + (1-xy) \wedge xy + (1-y) \wedge (y-xy) \\
& - (1-xy) \wedge (y-xy) - (1-y) \wedge (1-xy)
\end{aligned}$$

Expanding and rearranging gives:

$$\begin{aligned}
=\ & \big(x \wedge (1-x) + (1-x) \wedge x\big) + \big(y \wedge (1-y) + (1-y) \wedge y\big) \\
& + \big((1-x) \wedge (1-y) + (1-y) \wedge (1-x)\big) + \big((1-xy) \wedge y - (1-xy) \wedge y\big) \\
& + \big(-(1-xy) \wedge x + (1-xy) \wedge x\big) - \big((1-x) \wedge (1-xy) + (1-xy) \wedge (1-x)\big) \\
& - \big((1-y) \wedge (1-xy) + (1-xy) \wedge (1-y)\big) \\
=\ & 0
\end{aligned}$$

□

The fact that the above defined element ρ is an element in $\ker \beta_2$ is fundamental to the definition of the Bloch group, which we now define. In the definition we mention the 'five-term relations', this refers to the fact that there are several forms of the five-term relation, all of which are in the kernel of the map β_2, at least up to possible 2-torsion.

Definition 5.2.5. *The* ***Bloch group*** *is defined to be the quotient group*

$$\frac{\ker \beta_2}{<\textit{five-term relations}>}.$$

Remark 5.2.6. *Unfortunately, there are more than one definition of the Bloch group. Some authors define the Bloch group to be the quotient group*

$$\frac{\mathbb{Z}[K^*]}{<\textit{five-term relations}>}$$

which has many more elements than our definition. We will use definition 5.2.5 for the purpose of this report.

So far we have only seen elements in the kernel of β_2 which are 'universal' elements; they are elements of $\ker \beta_2$ irrespective of which algebraic number field we are working in. These elements are trivial Bloch group elements. We would imagine that since elements in an algebraic number field can satisfy specific functional equations unique to that algebraic number field, that we can utilise these to find elements in the Bloch group unique to the algebraic number field. A very simple example follows.

Example 5.2.7. *We look at the n-th cyclotomic field, $K = \mathbb{Q}(\theta)$, where $\theta^n = 1$. Since we have the following property of the wedge product*

$$c(a \wedge b) = a^c \wedge b.$$

We can easily formulate an element of the Bloch group, namely $\rho = n[\theta]$, since

$$\begin{aligned} n[\theta] &\longmapsto n(\theta \wedge (1-\theta)) \\ &= \theta^n \wedge (1-\theta) \\ &= 1 \wedge (1-\theta) \\ &= 0. \end{aligned}$$

We now explain how we can use the Bloch group and Bloch-Wigner function to find 'higher' regulators.

5.3 Obtaining the Higher Regulator

Our aim is that together, the Bloch group and the Bloch-Wigner function will formulate a 'higher' regulator for the algebraic number field. We will call this higher regulator reg^2_K, the 2 signifying that we are in the new higher case which we have denoted case $m = 2$. We will now state explicitly how this can be obtained.

To interpret our new regulator as being a higher case of the Dirichlet regulator already constructed, we look to mimic the construction of the Dirichlet regulator when constructing reg^2_K. In essence, to obtain the Dirichlet regulator, R_K of an algebraic number field K, we worked through the following steps:

1. Find the non-torsion units of K.
2. Find the different embeddings into $\mathbb{C}$ of these units.
3. Take the logarithm of these embeddings.
4. Form a matrix out of these values.
5. Take the determinant of a maximum minor of this matrix.

We now replicate this process in a new higher case. So to find reg^2_K we follow a very similar process to the above, it is very easy to see how each of the steps above pair up with their 'analogy' below. The process is given:

1. Find 'non-standard' elements of the Bloch group of K.

2. Find the different embeddings into $\mathbb{C}$ of these elements.

3. Apply the Bloch-Wigner function to these values.

4. Form a matrix out of these values.

5. Take the determinant of this matrix.

However, there are still questions to be answered as to how we will actually carry out this process. For example, how many elements in the Bloch group do we need to find? We will come to these answers shortly, but first we will give an outline of the theory as to why the Bloch-Wigner function and the Bloch group are used in the above process.

5.3.1 Volumes of Hyperbolic 3-Manifolds

The title of this section at first appears to be completely incongruent with the current flow of this report. However, it is via evaluations of the volume of hyperbolic 3-manifolds that we manage to link the Dedekind zeta function to the Bloch-Wigner function. What follows is a brief outline and not a definitive description of this process. The main source of this material is the article [23], which goes into more detail. The reader is assumed to have an idea to the workings of hyperbolic geometry. We will first look at hyperbolic 2-space as it provides a good analogy and is easier to visualise.

Hyperbolic 2-Space

We consider the Poincaré half-plane model of hyperbolic 2-space, denoted $\mathbb{H}^2$. This is a conformal model which, by distorting distances, allows us to view $\mathbb{H}^2$ as a Euclidean space. The model maps $\mathbb{H}^2$ on to an upper half plane, often thought of to be $\mathbb{C}$ with $\Im(z) \geq 0$, where distances are distorted by a metric which becomes more dense as we approach the bottom of the half-plane.

It is well known that the geodesics of $\mathbb{H}^2$ on the half-plane model are all the semicircles centred on the bottom axis and the vertical straight lines (as shown in the figure 5.1).

Definition 5.3.1. *A* ***Möbius transformation*** *is a conformal (or angle preserving) map from and to the complex plane, of the form*

$$z \longmapsto \frac{az+b}{cz+d} \quad \textit{such that} \quad ad-bc \neq 0,\ z,a,b,c,d \in \mathbb{C}.$$

A Möbius transformation can be viewed as an element of the general linear group, $GL(2,\mathbb{C})$, the set of all 2x2 matrices with non-zero determinant. This is done by noting

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} z \\ 1 \end{pmatrix} = \begin{pmatrix} az+b \\ cz+d \end{pmatrix}.$$

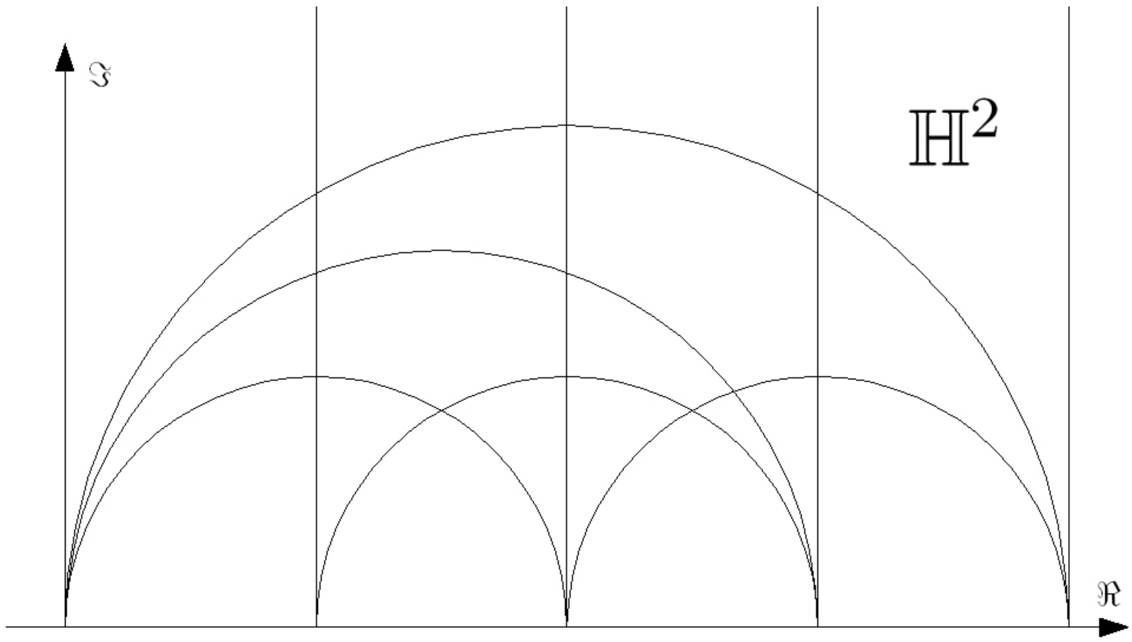

Figure 5.1: The Poincaré half-plane model of $\mathbb{H}^2$ with some example geodesics

By dividing the top coordinate by the lower coordinate we obtain the result of a Möbius transformation. This is similar to the way we construct projective coordinates.

We now examine how a normal subgroup of $GL(2, \mathbb{C})$, namely the special linear group on the integers, $SL_2(\mathbb{Z})$, acts on elements of the hyperbolic plane. The special linear group on the integers is defined by

$$SL(2,\mathbb{Z}) := \left\{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \det(A) = 1 \right\}.$$

We are particularly interested in the action of the elements

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in SL(2,\mathbb{Z}).$$

We see that

$$\begin{aligned} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} z \\ 1 \end{pmatrix} &= \begin{pmatrix} 1+z \\ 1 \end{pmatrix} \\ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} z \\ 1 \end{pmatrix} &= \begin{pmatrix} -1 \\ z \end{pmatrix}. \end{aligned}$$

So the elements relate to the Möbius transformations $t(z)$ and $i(z)$ given by

$$t : z \longmapsto z + 1$$

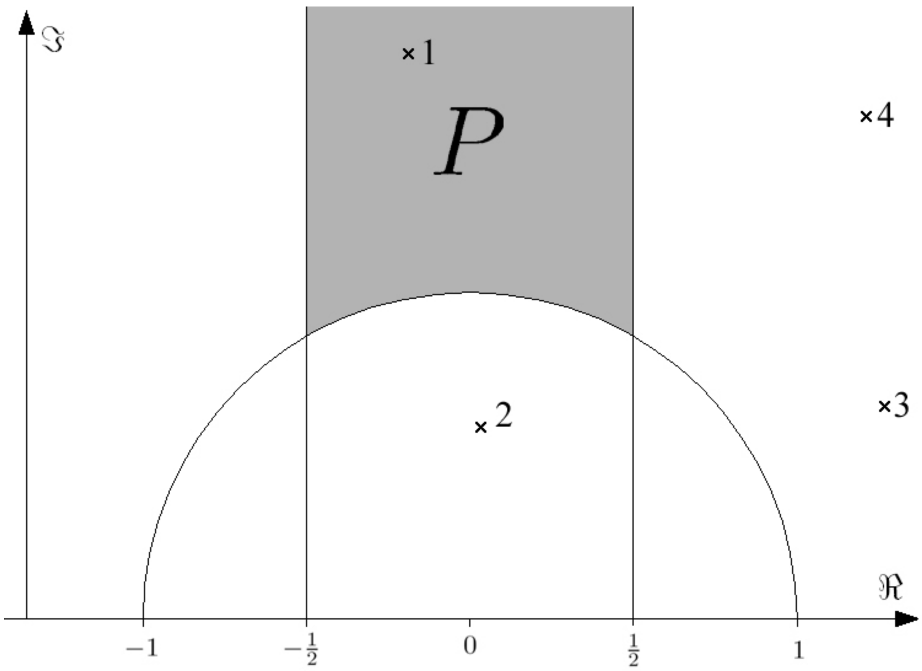

Figure 5.2: Fundamental Domain of $\mathbb{H}^2$ under transformations in $SL(2, \mathbb{Z})$ with the four cases from the proof of proposition 5.3.2.

and

$$i : z \longmapsto \frac{-1}{z}.$$

These transformations are the translation by 1 in the real direction and inversion in the unit circle around the origin respectively.

Proposition 5.3.2. *Let*

$$P = \left\{ z \in \mathbb{H}^2 \;\middle|\; -\frac{1}{2} < z < \frac{1}{2}, \, |z| \geq 1 \right\}$$

shown in figure 5.2.

Any element $v \in \mathbb{H}^2$ can be conformally mapped to an element in P using a sequence of applications of the maps

$$t : z \longmapsto z + 1$$

and

$$i : z \longmapsto \frac{-1}{z}$$

or their inverses.

Proof. We provide an idea of a proof of this, based on [12]. There are four possible situations:

1. $v \in P$.
2. $\Im(v) < 1$ and $-\frac{1}{2} \leq \Re(z) \leq \frac{1}{2}$.
3. $\Im(v) < 1$ and $\Re(z) < -\frac{1}{2}$ or $\Re(z) > \frac{1}{2}$.
4. $\Im(v) \geq 1$ and $\Re(z) < -\frac{1}{2}$ or $\Re(z) > \frac{1}{2}$.

Possible examples of these four cases are shown in figure 5.2 with the labels reflecting the numbered cases above.

By applying the map $t(z)$, or its inverse, a finite number of times, any point in situation 3 or 4 becomes a point in situation 1 or 2. Since we require every point to be mapped to a point in situation 1 under a combination of maps $t(z)$ and $i(z)$, all that remains is to prove that any point in situation 2 can be mapped to a point in situation 1.

Taking v to be a point such that $\Im(v) < 1$ and $-\frac{1}{2} < z < \frac{1}{2}$ we see that the point $i(v)$ will either be in situation 1, 3 or 4. Again, if we are now in situation 1 we are done. If we are in situation 3 or 4 we apply the transformation $t(z)$ until we are left with a point in situation 1 or 2. We now hypothesise that by repeatedly applying the maps $i(z)$ and $t(z)$ to a point in situation 2, we always result in a point in situation 1 in a finite number of steps. We can prove this hypothesis using the following two facts, given without proof.

- For $v \in \mathbb{H}^2$, such that $-\frac{1}{2} \leq \Re(z) \leq \frac{1}{2}$ and $0 < \Im(v) \leq 1$, then
$$\Im(-i(v)) \geq 2\Im(v).$$
- For $v \in \mathbb{H}^2$, such that $-\frac{1}{2} \leq \Re(z) \leq \frac{1}{2}$ and $|v| < 1$ and $\Im(v) > \frac{1}{2}$ then
$$|i(v)| > 1,\ |t^{-1}(i(v))| \geq 1,\ |t(i(v))| \geq 1,\ -1 \leq \Re(i(v)) \leq 1$$

We do not give the full proof of this proposition here as it is rather long and is not important for this report. □

We conclude that P is a fundamental domain of the space $\mathbb{H}^2$ under transformations in $SL(2,\mathbb{Z})$. We can also view elements in P as elements in the quotient group $\mathbb{H}^2/SL(2,\mathbb{Z})$.

Linking $\zeta_K(2)$ and $D(z)$ via Hyperbolic 3-Space

The following description is fairly brief as this is beyond the scope of this report. However, it enables the reader to get a feel of the idea and workings behind a very important link. We restrict ourselves to only examining imaginary quadratic fields K. An imaginary quadratic field is a field $K = \mathbb{Q}(\theta)$ with $\theta^2 = -d$ and $1 \neq d \in \mathbb{Z}$, for a squarefree d. We move to hyperbolic 3-space and look at the the quotient group

$$\frac{\mathbb{H}^3}{SL(2, \mathcal{O}_K)}.$$

We can view hyperbolic 3-space in a similar way to hyperbolic 2-space, where instead of the base being a line, it is now a plane. We can view this as the complex plane. We find that the geodesics in this space are semispheres and planes. We are interested in quotienting out the space by the special linear group over the ring of integers of an algebraic number field. This results in a fundamental domain, P, which bears some similarities to the one we obtained before. To give an idea of the possible fundamental domain that could be obtained, figure 5.3 is a representation of how the fundamental domain of $\mathbb{H}^3$ under transformations related to a certain subgroup of

$$SL\left(2, \mathbb{Z}\left[\frac{1+\sqrt{-3}}{2}\right]\right),$$

may look. In this case P is in fact a hyperbolic tetrahedron.

The importance of studying hyperbolic space becomes very relevent to our goal when we observe how the volume of this fundamental domain links to both the Dedekind zeta function and the Bloch-Wigner function. This astonishing link is at the heart of our aim to formulate a higher analogy of the analytic class number formula. The following relation, credited to Humbert, is the link from the Dedekind zeta function to the volume of this fundamental domain, which we state without proof (a proof can be found in [9]).

Theorem 5.3.3. *(Humbert's volume formula) For an algebraic number field K, with discriminant d_K, and $\zeta_K(s)$ the Dedekind zeta function for K, then*

$$\zeta_K(2) = \frac{4\pi^2}{d_K\sqrt{d_K}} Vol\left(\frac{\mathbb{H}^3}{SL(2, \mathcal{O}_K)}\right).$$

The link between the Bloch-Wigner function and the volume of this fundamental domain comes from an important property of the Bloch-Wigner function. The volume of a tetrahedron in hyperbolic 3-space with vertices at 0, 1, z and ∞, shown in figure 5.4, is exactly equal to $D(z)$. In fact, a general hyperbolic tetrahedron can be calculated from a combination of 24 values of the Bloch-Wigner function. This would allow us, for example, to calculate the volume of the tetrahedron in figure 5.3, in as a sum of Bloch-Wigner values. Most other algebraic number fields will not result in tetrahedron fundamental domains, but these can be triangulated to give a sum of volumes of tetrahedra. We invite the reader to see [18] and chapter 1 of [6] for a more detailed descriptions of how this is achieved.

It is possible to triangulate the fundamental domain of the group $\mathbb{H}^3/SL(2, \mathcal{O}_K)$ into tetrahedra whose volume can be calculated using the Bloch-Wigner function applied to elements of $\mathcal{O}_K$. A condition that arises from this process is that the triangulation forces the elements to be

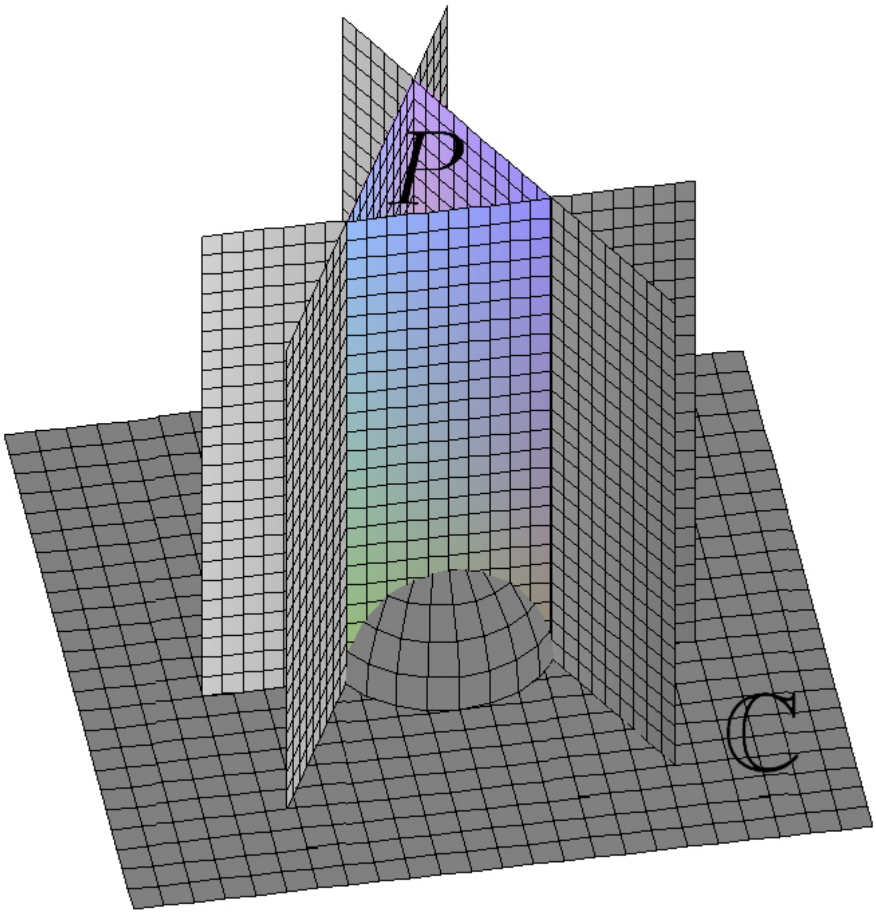

Figure 5.3: Diagram showing a possible fundamental domain, P, of $\mathbb{H}^3$ under transformations related to a certain subgroup of $SL\left(2, \mathbb{Z}\left[\frac{1+\sqrt{-3}}{2}\right]\right)$. P is the region bounded by the three planes and above the unit sphere.

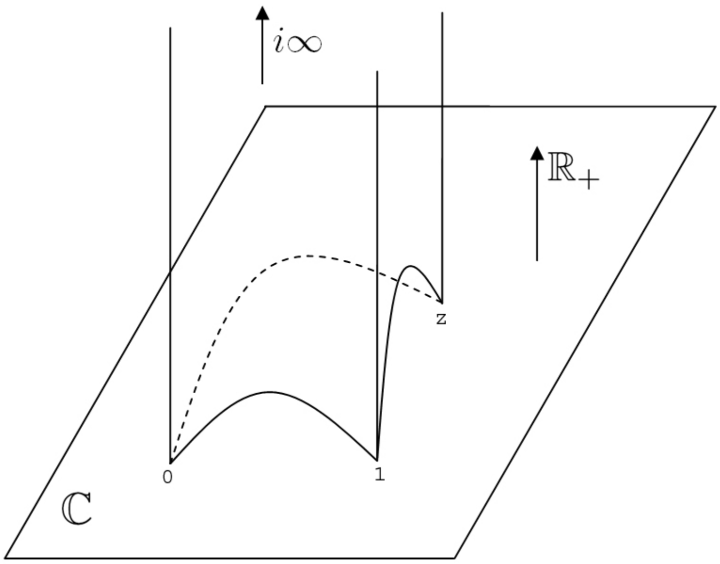

Figure 5.4: A diagram showing a tetrahedron in $\mathbb{H}^3$ with volume $D(z)$

elements of the Bloch group we constructed earlier. For details of the triangulation and its implications, see [23] and also [24].

Conclusion

The above evidence gives us good reason to believe that we can secure a link between the Dedekind zeta value of an algebraic number field, K, and the Bloch-Wigner Function via the volume of a fundamental domain in $\mathbb{H}^3$ under transformations by elements of $SL(2, \mathcal{O}_K)$. This is the Zagier Conjecture, which follows.

5.4 Zagier's Conjecture

We now state Zagier's Conjecture, the highlight of the second part of this report. It is taken from Zagier's own paper, [21], but slightly modified to suit the notation used in this report.

Theorem 5.4.1. *Let K be an arbitrary algebraic number field, d_K the discriminant of K, r_1 and r_2 the numbers of real and pairs of complex places, and the Dedekind zeta function $\zeta_K(s)$. Then*

$$\zeta_K(2) = \frac{\pi^{2(r_1+r_2)}}{\sqrt{|d_K|}}\Phi.$$

Where Φ is a rational linear combination of r_2-fold products $D(\sigma_{r_1+1}(\alpha))\cdots D(\sigma_{r_1+r_2}(\alpha))$ with $\alpha \in K$ and where the σ_i are the various embeddings from $K \hookrightarrow \mathbb{C}$ described in chapter 2.

We will focus on the term Φ in the above equation, very nearly the 'higher' regulator, which we have denoted reg_K^2. The multiplicative difference is the 'higher' analogues of the values h_K and w_K, which we obtain from algebraic K-theory. As stated before, we will not cover this here. This multiplicative difference is however, always rational, which we will use when making some specific evaluations later.

Notation 5.4.2. *When writing an element, β, of the we have written them in the form*

$$\sum_{i=1}^{c} a_i[\beta_i].$$

Where $a_i, c \in \mathbb{Z}$ and $\beta_i \in \mathcal{O}_K$. What we will mean by 'applying the Bloch-Wigner function to the element β' will be the following

$$\sum_{i=1}^{c} a_i D(\sigma(\beta_i)).$$

Where the σ is a specified embedding $K \hookrightarrow \mathbb{C}$.

We find reg_K^2 by taking the determinant of a matrix formed out of Bloch-Wigner values of embeddings of r_2 elements of the Bloch group. This is the rational linear combination of r_2-fold products from the Zagier conjecture (ignoring the term just mentioned). The matrix will take the form

$$\begin{pmatrix} D(\sigma_{r_1+1}(\alpha_1)) & \dots & D(\sigma_{r_1+r_2}(\alpha_1)) \\ \vdots & \dots & \vdots \\ D(\sigma_{r_1+1}(\alpha_{r_2})) & \dots & D(\sigma_{r_1+r_2}(\alpha_{r_2})) \end{pmatrix},$$

where the $\alpha_1, \dots \alpha_{r_2}$ are elements in the Bloch group of K, the algebraic number field. However, we ask, which elements of the Bloch group are these and how do we find them?

5.4.1 Finding Suitable Bloch Group Elements

Finding r_2 elements in the Bloch group which enable us to calculate a non-zero determinant in the matrix described above is very difficult. As mentioned before, we have elements of the Bloch group which are formed out of functional equations of the Bloch-Wigner function and elements $x, y \in K$, such as

- $[x] + [y] + \left[\frac{1-x}{1-xy}\right] + [1-xy] + \left[\frac{1-y}{1-xy}\right]$

and

- $[x^N] - N \sum_{\zeta^N=1} [\zeta x] \in \ker \beta_2$ for $N \in \mathbb{Z}_{\geq 0}$.

However, since we know that we get zero when we apply the Bloch-Wigner function, it is obvious that if these were used in the matrix we require we would get a determinant of zero. We suspect that a 'useful element' will be in the Bloch group due to a property specific to the algebraic number field, such as in Example 5.2.7. However, even if r_2 elements of this form are found; the resulting determinant may still be zero. We will not explain explicitly how these elements can be located, however we will find elements in the Bloch group which enable us to calculate the Dedekind zeta value at 2 for a simple algebraic number field. This will serve to give an idea of process.

5.4.2 Applying the Reflection Property of the Dedekind zeta Function

In the previous chapter we simplified the analytic class number formula by using the reflection property of the dedekind zeta function (see section 4.1.3). The Dedekind zeta function, $\zeta_K(s)$, has a pole at $s = 1$ and a zero at $s = 0$. We calculated the residues of these values for the analytic class number formula

$\zeta_K(2)$, however, is a finite non-zero value and we can simply calculate the value using a computer program such as GP/PARI [8]. However, to reflect the previous chapter, we can use the reflection property of the Dedekind zeta function on evaluations to express 'values' of negative Dedekind zeta values in terms of the Bloch-Wigner function. The Dedekind zeta function has zeros at the negative integers of order r_2 if s is odd and of order $r_1 + r_2$ if s is even. In the previous chapter we denoted by $\zeta_K^*(0)$ the residue of the zero at $s = 0$ of the Dedekind zeta function. We extend this definition to integers $s < 0$.

Definition 5.4.3. *We define $\zeta_K^*(t)$ for $t \in \mathbb{Z}^{<0}$ as follows*

$$\zeta_K^*(t) = \begin{cases} \lim_{s \to t} \dfrac{\zeta_K(s)}{(s-t)^{r_1+r_2}} & \text{if } t \in \mathbb{Z}^{<0},\ t \text{ even} \\[2ex] \lim_{s \to t} \dfrac{\zeta_K(s)}{(s-t)^{r_2}} & \text{if } t \in \mathbb{Z}^{<0},\ t \text{ odd.} \end{cases}$$

From here we see that, using the reflection property of the Dedekind zeta function and an evaluation of $\zeta_K(2)$ as outlined in Zagier's conjecture, it is possible to express the value $\zeta_K^*(-1)$ in terms of the Bloch-Wigner function of Bloch group elements.

5.5 $\zeta_K(2)$ for Cyclotomic Fields

As in Chapter 3 we will look at cyclotomic fields to make specific evaluations. We start with the seventh cyclotomic field.

5.5.1 $\zeta_K(2)$ for the Seventh Cyclotomic Fields

In chapter 3 we found the Dirichlet regulator, R_K, where K was the seventh cyclotomic field. From this we found, in a sense, the value of $\zeta_K(s)$ at $s = 1$, where we in fact found the residue of the pole at this point. For this we used the analytic class number formula. We now look for the value $\zeta_K(2)$ using Zagier's Conjecture.

Due to the properties of this field it is easy to find non-trivial Bloch group elements. In fact, the elements we will use are the elements from Example 5.2.7. For the seventh cyclotomic field we denote by θ the root of the minimum polynomial of K, which is $x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$. From Example 5.2.7 we find that $7[\theta]$ is an element of the Bloch group. In fact, $7[\theta^n]$ for $n = 1, \ldots, 6$ are also elements of the Bloch group. We require r_2 elements, so we will take $n = 1, 2, 4$ and obtain

$$\alpha_1 = 7[\theta], \alpha_2 = 7[\theta^2] \text{ and } \alpha_3 = 7[\theta^4]$$

as our Bloch group elements. These elements are independent, we will not describe how to show these are in fact independent elements here but invite the reader to see [23] and [21].

Remark 5.5.1. *To maximise our chances of finding elements of the Bloch group for an algebraic number field we try elements that are distinct and are not universal elements of the kernel of β_2.*

Using the same notation as in chapter 3 for the σ_i, we form the matrix described in the previous section and denote this matrix

$$\begin{aligned} A &= \begin{pmatrix} 7D(\sigma_1(\theta)) & 7D(\sigma_2(\theta)) & 7D(\sigma_3(\theta)) \\ 7D(\sigma_1(\theta^2)) & 7D(\sigma_2(\theta^2)) & 7D(\sigma_3(\theta^2)) \\ 7D(\sigma_1(\theta^4)) & 7D(\sigma_2(\theta^4)) & 7D(\sigma_3(\theta^4)) \end{pmatrix} \\ &= 7^3 \begin{pmatrix} D(\zeta_7) & D(\zeta_7^2) & D(\zeta_7^3) \\ D(\zeta_7^2) & D(\zeta_7^4) & D(\zeta_7^6) \\ D(\zeta_7^4) & D(\zeta_7) & D(\zeta_7^5) \end{pmatrix}. \end{aligned}$$

In Appendix section B.3.1 we use the program GP/PARI [8] to find this matrix numerically. We then take the determinant and arrive at a numerical approximation for reg_K^2

$$\mathrm{reg}_K^2 = \det(A) = 7^3 \cdot 2.3150777443781813951322407 94$$

Since the discriminant of the p-th cyclotomic field is equal to $p^{(p-2)}$ and for the 7-th cyclotomic field, $r_1 = 0$ and $r_2 = 3$ then from Zagier's conjecture we expect

$$\frac{\pi^{2(r_1+r_2)}}{\sqrt{|d_K|}}\mathrm{reg}_K^2 = \frac{\pi^6}{\sqrt{7^5}}\mathrm{reg}_K^2$$

to be rationally equivalent to $\zeta_K(2)$. Or, in other words, that

$$\frac{7^5 \cdot \zeta_K(2)}{\pi^6 \cdot \mathrm{reg}_K^2} \in \mathbb{Q}.$$

This is numerically evaluated in Appendix section B.3.1 and denoted `zetak72/rhs7`. Using the GP/PARI function `lindep` we can check the linear dependence of this number with 1 and we confirm numerically that this value is indeed an element of the rational numbers, in fact

$$\frac{7^5 \cdot \zeta_K(2)}{\pi^6 \cdot \mathrm{reg}_K^2} \in \mathbb{Q} = \frac{352947}{64} = \frac{3 \cdot 7^6}{2^6}.$$

Since this is a relatively simple fraction, this is a convincing way to approximate $\zeta_K(2)$.

We can now state $\zeta_K(2)$, for the seventh cyclotomic field, in terms of the Bloch-Wigner function of elements of the Bloch Group, up to numerical confirmation.

$$\begin{aligned}
\zeta_K(2) &= \frac{2^6}{3 \cdot 7^6} \cdot \frac{\pi^6}{\sqrt{7^5}} \cdot \mathrm{reg}_K^2 \\
&= \frac{2^6 \pi^6}{3 \cdot 7^8 \sqrt{7}} \cdot 7^3 \det \begin{pmatrix} D(\zeta_7) & D(\zeta_7^2) & D(\zeta_7^3) \\ D(\zeta_7^2) & D(\zeta_7^4) & D(\zeta_7^6) \\ D(\zeta_7^4) & D(\zeta_7) & D(\zeta_7^5) \end{pmatrix} \\
&= \frac{2^6 \pi^6}{3 \cdot 7^5 \sqrt{7}} \cdot \det \begin{pmatrix} D(\zeta_7) & D(\zeta_7^2) & D(\zeta_7^3) \\ D(\zeta_7^2) & D(\zeta_7^4) & D(\zeta_7^6) \\ D(\zeta_7^4) & D(\zeta_7) & D(\zeta_7^5) \end{pmatrix} \\
&\approx 1.0677862284146970194463505531.
\end{aligned}$$

5.5.2 Generalising for Other Cyclotomic Fields

Using the method used for the above calculation of the value $\zeta_K(2)$ when K is the seventh cyclotomic field, we can easily generalise our working on GP/PARI for any cyclotomic field. The code for this is shown in Appendix section B.3.2. The following other expressions and evaluations are obtained using this method.

For K the 3-rd cyclotomic field, with ζ_3 the primitive third root of unity

$$\begin{aligned}
\zeta_K(2) &= 3\sqrt{3} \cdot \pi^2 \cdot D(\zeta_3) \\
&\approx 1.2851909554841494029175117 99.
\end{aligned}$$

For K the 5-th cyclotomic field, with ζ_5 the primitive fifth root of unity

$$\begin{aligned} \zeta_K(2) &= \frac{3 \cdot 5^2 \sqrt{5} \cdot \pi^4}{2^3} \cdot \det \begin{pmatrix} D(\zeta_5^2) & D(\zeta_5^4) \\ D(\zeta_5) & D(\zeta_5^2) \end{pmatrix} \\ &\approx 1.092349661730969782396547812. \end{aligned}$$

And for K the 11-th cyclotomic field, with ζ_{11} the primitive eleventh root of unity

$$\begin{aligned} \zeta_K(2) &= \frac{3 \cdot 11^5 \sqrt{11} \cdot \pi^{10}}{2^1 2} \cdot \det B \\ &\approx 1.03209498358 \end{aligned}$$

Where:

$$B = \begin{pmatrix} D(\zeta_{11}^9) & D(\zeta_{11}^7) & D(\zeta_{11}^5) & D(\zeta_{11}^3) & D(\zeta_{11}) \\ D(\zeta_{11}^{10}) & D(\zeta_{11}^9) & D(\zeta_{11}^8) & D(\zeta_{11}^7) & D(\zeta_{11}^6) \\ D(\zeta_{11}^5) & D(\zeta_{11}^{10}) & D(\zeta_{11}^4) & D(\zeta_{11}^9) & D(\zeta_{11}^3) \\ D(\zeta_{11}^7) & D(\zeta_{11}^3) & D(\zeta_{11}^{10}) & D(\zeta_{11}^6) & D(\zeta_{11}^2) \\ D(\zeta_{11}^8) & D(\zeta_{11}^5) & D(\zeta_{11}^2) & D(\zeta_{11}^{10}) & D(\zeta_{11}^7) \end{pmatrix}.$$

Chapter 6

Conclusion and Outlook

6.1 Summary

This report has concerned the first two cases of a grander theory. Chapters 2, 3 and 4 concern the case $m = 1$, or the link between a Dedekind zeta value of an algebraic number field at 1 and the logarithm of representations of units. While chapter 5 concerns the case $m = 2$; the link between the Dedekind zeta value at 2 and the Bloch-Wigner function. We now summarise these results.

6.1.1 Case $m = 1$

For an algebraic number field K, the analytic class number formula provides us with a link between the residue of the pole $s = 1$ of $\zeta_K(s)$ and almost all of the important invariants of K. One version of the formula reads

$$\mathrm{res}_{s=1}\zeta_K(s) = \frac{2^{r_1+r_2}\pi^{r_2}h_K R_K}{w_K\sqrt{|d_K|}}$$

where

- h_K is the class number of K, an important invariant which measures to what extent unique factorisation fails in K.
- w_K is the number of roots of unity contained in K.
- R_K is the Dirichlet Regulator.
- d_K is the discriminant of K.

We focused on the term R_K, the Dirichlet regulator . This was explicitly constructed in chapter 2 and is summarised by the following steps:

1. Construct geometric representation for an algebraic number field in a subspace of $\mathbb{R}^k$.
2. Move this representation into a logarithmic space, again a subspace of $\mathbb{R}^k$
3. Find the coordinates for units in a system of $r_1 + r_2 - 1$ fundamental units.
4. Find R_K by either
    - Finding the volume of a fundamental parallelepiped of the lattice formed by representations of the units. Then dividing this value by $\sqrt{r_1 + r_2}$.

   or

    - Forming a matrix from the vectors representing the system of fundamental units and calculating the determinant of a maximum minor.

When we examine the analytic class number formula we see that it links the the Dedekind zeta value of 1 with the logarithm, which is, in essence, the first polylogarithm. From this we motivated moving to consider 'higher' cases with 'higher' Dedekind zeta values and 'higher' logarithms.

6.1.2 Case $m = 2$

In a similar fashion to the case $m = 1$, a formula provides us with a link between the Dedekind zeta value at 2 and the dilogarithm of 'higher' units. The formula in question is the one formed in Zagier's conjecture

$$\zeta_K(2) = \frac{\pi^{2(r_1+r_2)}}{\sqrt{|d_K|}}\Phi.$$

Where the value $\Phi = c \cdot \mathrm{reg}_K^2$ with $c \in \mathbb{Q}$. The value c is obtained from algebraic K-theory and was not discussed in this report and represents 'higher' values of h_K and w_K. The value reg_K^2 is the 'higher' regulator which can be found using the following steps, often with difficulty:

1. Find r_2 independent, non-torsion elements in the Bloch group of K.
2. Apply the Bloch-Wigner function to r_2 embeddings of each of the r_2 independents Bloch group elements and form a matrix from these values.
3. The determinant of this matrix gives us reg_K^2.

Finding the necessary independent non-torsion Bloch elements is very difficult. For this report we found the required elements in cyclotomic fields, which are easier to find and are in fact fairly simple. Finding these elements for quadratic fields, $\mathbb{Q}(\sqrt{-a})$ can also be achieved, but gets very complicated very quickly as we increase $a \in \mathbb{Z}$. From [23] we see that even for the algebraic number field $K = \mathbb{Q}(\sqrt{-23})$ a Bloch element already is fairly complicated. It is

$$21\left[\frac{1+\sqrt{-23}}{2}\right] + 7[2+\sqrt{-23}] + 3\left[\frac{3+\sqrt{-23}}{2}\right] - 3\left[\frac{5+\sqrt{-23}}{2}\right] + [3+\sqrt{-23}].$$

Formulating Bloch elements for more complicated algebraic number fields is a very difficult task.

6.2 Moving Even Higher

The similarities between the analytic class number formula and Zagier's conjecture are obvious, and this is strong motivation for taking the whole idea higher. The next step is to find the Dedekind zeta value at 3 in terms of the trilogarithm. An article on this topic, by Don Zagier, is [22].

We will not discuss this topic in this report, however, as a small taster we make one definition. To take it to the next case, there is a need to find a higher alternative to the Bloch-Wigner function, namely the Bloch-Wigner-Ramakrishnan function, defined implicitly by Ramakrishnan in [13], however from [21] by Don Zagier we give the following explicit definition.

Definition 6.2.1. *The **Bloch-Wigner-Ramakrishnan function** is a single valued function for $z \neq 0, \infty$ where $z \in \mathbb{C}$ and is defined seperately for cases $|z| \leq 1$ and $|z| > 1$.*

- *For $|z| \leq 1$*

$$D_m(z) = \Re_m\left(\Sigma_{r=0}^{m-1}\frac{(-1)^r}{r!}\ln^r|z|Li_{m-r}(z) - \frac{(-1)^m}{2m!}\ln^m|z|\right).$$

Where $\Re_m = \Re$ when m is odd and $\Re_m = \Im$ when m is even.

- *For $|z| > 1$*

$$D_m(z) = (-1)^{1-m}D_m\left(\frac{1}{z}\right).$$

The hope of this line of research is to be able to formulate the Dedekind zeta value of any integer using polylogarithms. The large complexity of even the second case implies that this undertaking will be an immensely difficult one, but the beauty of the current findings is good enough motivation to persevere.

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Bibliography

[1] Maplesoft. a division of Waterloo Maple Inc. Maple 11.0. Computer Software, 2007.

[2] Victor Abrashkin. Galois Theory. Lecture course given at Durham University, 2006.

[3] K Belabas. Pari-gp reference card. Adobe Systems Portable Document Format available at http://pari.math.u-bordeaux.fr/pub/pari/manuals/2.1.6/refcard.pdf, 2006.

[4] Z.I. Borevich and I. R. Shafarevich. *Number Theory.* Academic Press, 1966.

[5] Pierre Cartier. An Introduction to Zeta Functions. *From Number Theory to Physics*, pages 1–63, 1992.

[6] Jürgen Elstrodt, Fritz Grunewald, and Jens Mennicke. *Groups Acting on Hyperbolic Space.* Springer-Verlag Berlin Heidelberg, 1998.

[7] Herbert Gangl. Number Theory. Lecture course given at Durham University, 2006.

[8] The PARI Group. Gp/pari. Computer Software, 2003-2008.

[9] Fritz Grunewald and Stefan Kühnlein. On the Proof of Humbert's Volume Formula. *Journal manuscripta mathematica, Issue Volume 95, Number 1*, pages 431–436, 1998.

[10] Richard M. Hain. Classical Polylogarithms. *Pure Math.*, 55:3–42, 1994.

[11] John Milnor. *Introduction to Algebraic K-Theory.* Princeton University Press, 1972.

[12] John Parker. Elliptic Functions. Lecture course given at Durham University, 2006.

[13] D Ramakrishnan. Analogs of the Bloch-Wigner Function for Higher Polylogarithms. *Contemp. Math 55*, pages 371–376, 1986.

[14] John R. Silvester. *Introduction to Algebraic K-theory.* CRC Press Inc., 1981.

[15] Harold M. Stark. Galois Theory, Algebraic Number Theory, and Zeta Functions. *From Number Theory to Physics*, pages 313–393, 1992.

[16] H.M. Stark. *The Analytic Theory of Algebraic Numbers.* Springer-Verlag, New York Inc., 1993.

[17] Ian Stewart. *Galois Theory 3rd Edition*. Chapman and Hall, 2000.

[18] W. Thurston. *The Geometry and Topology of 3-Manifolds*, chapter 7. Princeton Univ. Mimeographed Notes.

[19] Lawrence C Washington. *Introduction to Cyclotomic Fields.* Springer-Verlag, New York Inc., 1982.

[20] Don Zagier. The Remarkable Dilogarithm. *J. Math and Phys. Soc. 22*, pages 131–145, 1988.

[21] Don Zagier. The Bloch-Wigner-Ramakrishnan Polylogarithm Function. *Math-Annalen 286*, pages 612–624, 1990.

[22] Don Zagier. Polylogarithms, Dedekind Zeta Functions, and the Algebraic K-Theory of Fields. *Arithmetic Algebraic Geometry, Prog. in Math. 89*, pages 391–430, 1991.

[23] Don Zagier and Herbert Gangl. Classical and Elliptic Polylogarithms and Special Values of L-Series. *The arithmetic and geometry of algebraic cycles (Banff, AB, 1998) NATO Sci. Ser. C Math. Phys. Sci., Kluwer Acad. Publ.*, pages 561–615, 2000.

[24] Don Zagier and W. D. Neumann. Volumes of Hyperbolic 3-Manifolds. *Topology*, 24:307–332, 1985.

Appendix A

Appendix

A.1 Properties of the Dilogarithm and the Bloch-Wigner Function

A.1.1 The Dilogarithm

"The dilogarithm is perhaps the only mathematical function with a sense of humo" - Don Zagier

After the logarithm, the simplest polylogarithm is the dilogarithm. As with the natural logarithm, the dilogarithm has many interesting properties.

Special Values

In [20] Don Zagier comments that as far as is known at this point, there are 8 values of the dilogarithm that can be expressed in closed form. They are expressed in terms of powers of π

and the natural logarithm as follows

$$
\begin{aligned}
\mathrm{Li}_2(0) &= 0 && \text{(A.1)}\\
\mathrm{Li}_2(1) &= \frac{\pi^2}{6} && \text{(A.2)}\\
\mathrm{Li}_2(-1) &= \frac{-\pi^2}{12} && \text{(A.3)}\\
\mathrm{Li}_2\left(\frac{1}{2}\right) &= \frac{\pi^2}{12} - \frac{1}{2}\ln^2(2) && \text{(A.4)}\\
\mathrm{Li}_2\left(\frac{3-\sqrt{5}}{2}\right) &= \frac{\pi^2}{15} - \ln^2\left(\frac{1+\sqrt{5}}{2}\right) && \text{(A.5)}\\
\mathrm{Li}_2\left(\frac{-1+\sqrt{5}}{2}\right) &= \frac{\pi^2}{10} - \ln^2\left(\frac{1+\sqrt{5}}{2}\right) && \text{(A.6)}\\
\mathrm{Li}_2\left(\frac{1-\sqrt{5}}{2}\right) &= -\frac{\pi^2}{15} - \frac{1}{2}\ln^2\left(\frac{1+\sqrt{5}}{2}\right) && \text{(A.7)}\\
\mathrm{Li}_2\left(\frac{-1-\sqrt{5}}{2}\right) &= -\frac{\pi^2}{10} - \frac{1}{2}\ln^2\left(\frac{1+\sqrt{5}}{2}\right) && \text{(A.8)}
\end{aligned}
$$

(A.1) is obvious from the definition of the dilogarithm, but (A.2) is immediately more interesting. By putting $z = 1$ into the definition of the dilogarithm we get

$$\mathrm{Li}_2(1) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \zeta(2)$$

where $\zeta(s)$ is the Riemann zeta function defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \quad s \in \mathbb{C}.$$

Lastly, the final four appear to relate to each other in their structure. We do not go into detail about these values here, but they hold some weight in this report out of curiosity. The curious thing here being the difficulty in finding values of the dilogarithm that can be expressed in closed form. As Don Zagier notes in [20], other similar functions, such as the Riemann zeta function, have many easy to find special values; not so with the humorous dilogarithm.

Functional Equations

The dilogarithm satisfies many functional equations. We state some of them here, taken from [20]. Firstly, the dilogarithm has two reflection formulas, one which relates a value to is reflection in the line $\mathrm{Re}(z) = \frac{1}{2}$ and one which relates a value to the inversion of the value with respect to the unit circle. Given respectively these are

$$\mathrm{Li}_2\left(\frac{1}{z}\right) = -\mathrm{Li}_2(z) - \frac{\pi^2}{6} - \frac{1}{2}\ln^2(-z) \tag{A.9}$$

and

$$\mathrm{Li}_2(1-z) = -\mathrm{Li}_2(z) + \frac{\pi^2}{6} - \ln(z)\ln(1-z) \tag{A.10}$$

These two functions enable us to relate the following six functions via elementary functions (i.e $\ln(z)$)

- $\mathrm{Li}_2(z)$
- $\mathrm{Li}_2\left(\frac{1}{1-z}\right)$
- $\mathrm{Li}_2\left(\frac{z-1}{z}\right)$
- $-\mathrm{Li}_2\left(\frac{1}{z}\right)$
- $-\mathrm{Li}_2(1-z)$
- $-\mathrm{Li}_2\left(\frac{z}{z-1}\right)$.

Example A.1.1. *The first and the second functions are equivalent modulo elementary function since*

$$\begin{aligned}
Li_2\left(\frac{1}{1-z}\right) &= -Li_2(1-z) - \frac{\pi^2}{6} - \frac{1}{2}\ln^2(z-1) \\
&= -\left(-Li_2(z) + \frac{\pi^2}{6} - \ln(z)ln(1-z)\right) - \frac{\pi^2}{6} - \frac{1}{2}\ln^2(z-1) \\
&= Li_2(z) - \frac{2\pi^2}{6} + \ln(z)\ln(1-z) - \frac{1}{2}\ln^2(z-1)
\end{aligned}$$

and $-\frac{2\pi^2}{6} + \ln(z)\ln(1-z) - \frac{1}{2}\ln^2(z-1)$ *is an elementary function.*

We can also find a functional equation for squaring the value, since

$$\mathrm{Li}(z^2) = 2(\mathrm{Li}_2(z) + \mathrm{Li}_2(-z)).$$

This generalises to the 'distribution' equation

$$\mathrm{Li}(x) = \sum_{z^n = x} \mathrm{Li}_2(z).$$

Finally, a very important relation is the two-variable, five-term relation discovered in many different forms by Spence (1809), Abel (1827), Hill (1828), Kummer (1840), Schaeffer(1846) and more. One form of the relation reads:

$$\begin{aligned}\mathrm{Li}_2(x) &+ \mathrm{Li}_2(y) + \mathrm{Li}_2\left(\frac{1-x}{1-xy}\right) + \mathrm{Li}_2(1-xy) + \mathrm{Li}_2\left(\frac{1-y}{1-xy}\right) \\ &= \frac{\pi^2}{2} - \ln(x)\ln(1-x) - \ln(y)\ln(1-y) + \ln\left(\frac{1-x}{1-xy}\right)\ln\left(\frac{1-y}{1-xy}\right)\end{aligned}$$

This can be obtained by applying and reapplying the relations $(A.9)$ and $(A.10)$.

A.1.2 The Transition to the Bloch-Wigner Function

The real beauty of the Bloch-Wigner function, and a hint at why it is sometimes more useful than the dilogarithm, comes when we replace the dilogarithm in the functional equations above with the Bloch-Wigner Function. Not only do these relations still hold, they become much simpler.

Firstly

$$\begin{aligned}D\left(\frac{1}{z}\right) &= -D(z), \\ D(1-z) &= -D(z).\end{aligned}$$

This naturally leads to the six functions that were originally equivalent modulo elementary equations, being equal, when we replace $\mathrm{Li}_2(z)$ with $D(z)$,

$$D(z) = D\left(\frac{1}{1-z}\right) = D\left(\frac{z-1}{z}\right) = D\left(\frac{1}{z}\right) = D(1-z)D\left(\frac{z}{z-1}\right).$$

This immediately leads to the two variable, five-term relation becoming

$$D(x) + D(y) + D\left(\frac{1-x}{1-xy}\right) + D(1-xy) + D\left(\frac{1-y}{1-xy}\right) = 0.$$

A.2 The Wedge Product

We define the wedge product of two elements within a field F to be

$$a \wedge b \qquad a, b \in F^*$$

where we allow the following

- **Formal Addition** - $(a \wedge b) + (c \wedge d) + \dots$

- **Bilinearity** - $a \wedge bc = (a \wedge b) + (a \wedge c)$ and $(ab \wedge c) = (a \wedge c) + (b \wedge c)$

and

- $a \wedge a = 0 \quad \forall a \,\epsilon\, F^*$

$(a \wedge b)$ is in fact an element of

$$\bigwedge^2 F^* \;:\; \frac{\mathbb{Z}[a \wedge b \mid a, b \,\epsilon\, F^*]}{< \text{Bilinearity and } a \wedge a = 0 >_{gp}}.$$

We now explore the wedge product and discover various identities, purely using the rules allowed.

Proposition A.2.1. $\forall a \,\epsilon\, F^*$ *we have*

1. $a \wedge 1 = 0 = 1 \wedge a$
2. $a \wedge b = -(b \wedge a)$
3. $a \wedge b = -(\frac{1}{a} \wedge b)$
4. $a^n \wedge b = n(a \wedge b) \quad \forall n \,\epsilon\, \mathbb{Z}$

Proof. 1. Using bilinearity we have that

$$\begin{aligned} a \wedge 1 = (a \wedge 1) + (a \wedge 1) \quad &\Rightarrow \quad (a \wedge 1) - (a \wedge 1) = a \wedge 1 \\ &\Rightarrow \quad a \wedge 1 = 0. \end{aligned}$$

2. Using bilinearity twice we have that

$$ab \wedge ab = (a \wedge a) + (b \wedge b) + (a \wedge b) + (b \wedge a) \quad \Rightarrow \quad a \wedge b = -(b \wedge a).$$

Since:

$$ab \wedge ab = a \wedge a = b \wedge b = 0.$$

3. Using identity 1 we have that

$$(a \wedge b) + \left(\frac{1}{a} \wedge b\right) = a * \frac{1}{a} \wedge b = (1 \wedge b) = 0 \quad \Rightarrow \quad a \wedge b = -\left(\frac{1}{a} \wedge b\right).$$

4. We can prove this by splitting $n \in \mathbb{Z}$ into three cases as follows
   (a) $n \geq 1$: Follows directly from the bilinearity.
   (b) $n = 0$: $a^n \wedge b = 1 \wedge b = 0$ from identity 1.
   (c) $n < 0$: Follows from applying identity 3 repeatedly.

$\square$

A very important map involving the wedge product is the following:

$$\begin{array}{rcl} \beta_2 : \mathbb{Z}[F^*] & \longrightarrow & \bigwedge^2 F^* \\ [x] & \longmapsto & x \wedge (1-x) \end{array}$$

The elements in the kernel of this map are of interest, as we see in section 5.2. Below are a few simple examples of elements in the kernel for $F = \mathbb{Q}$ with verification.

Example A.2.2.
- $2[-1] \longmapsto 2(-1 \wedge 2) = (-1)^2 \wedge 2 = 1 \wedge 2 = 0$
- $[3] + [2] \longmapsto (3 \wedge -2) + (-2 \wedge 3) = (3 \wedge -2) - (3 \wedge -2) = 0$
- $[-8] - 6[-2] \longmapsto (-8 \wedge 9) - 6(-2 \wedge 3)$ *Where:*

$$\begin{aligned} (-8 \wedge 9) - 6(-2 \wedge 3) &= (2 \wedge 9) + (2 \wedge 9) + (-2 \wedge 9) \\ &\quad -6(-2 \wedge 3) \\ &= 4(2 \wedge 3) + 2((2 \wedge 3) + (-1 \wedge 3)) \\ &\quad -6((2 \wedge 3) + (-1 \wedge 3)) \\ &= 6(2 \wedge 3) - 6(2 \wedge 3) \\ &= 0 \end{aligned}$$

Remark A.2.3. *The first of the above examples is an example of an element being in the kernel of β_2 up to 2-torsion. So we can say that $[-1]$ is an element of $\ker \beta_2$ up to 2-torsion.*

Appendix B

Program Code

B.1 Calculating R_K for the Specific Example in Chapter 3

```
> solve(x^6+x^5+x^4+x^3+x^2+x+1 = 0);
```

$$\cos\left(\frac{2}{7}\pi\right)+i\sin\left(\frac{2}{7}\pi\right),-\cos\left(\frac{3}{7}\pi\right)+i\sin\left(\frac{3}{7}\pi\right),-\cos\left(\frac{1}{7}\pi\right)+i\sin\left(\frac{1}{7}\pi\right),$$
$$-\cos\left(\frac{1}{7}\pi\right)+-i\sin\left(\frac{1}{7}\pi\right),-\cos\left(\frac{3}{7}\pi\right)+-i\sin\left(\frac{3}{7}\pi\right),\cos\left(\frac{2}{7}\pi\right)-i\sin\left(\frac{2}{7}\pi\right)$$

```
> seventhroot := cos((2/7)*Pi)+I*sin((2/7)*Pi);
```

$\cos\left(\frac{2}{7}\pi\right)+i\sin\left(\frac{2}{7}\pi\right)$

```
> A(a,b,c,d,e,f) :=Matrix([[a,-f,f-e,e-d,d-c,c-b],
[b,a-f,-e,f-d,e-c,d-b],[c,b-f,a-e,-d,f-c,e-b],[d,c-f,b-e,a-d,-c,f-b],
[e,d-f,c-e,b-d,a-c,-b],[f,e-f,d-e,c-d,b-c,a-b]]);
```

$$(a,b,c,d,e,f)\quad -> \quad \begin{bmatrix} a & -f & f-e & e-d & d-c & c-b \\ b & a-f & -e & f-d & e-c & d-b \\ c & b-f & a-e & -d & f-c & e-b \\ d & c-f & b-e & a-d & -c & f-b \\ e & d-f & c-e & b-d & a-c & -b \\ f & e-f & d-e & c-d & b-c & a-b \end{bmatrix}$$

```
> FieldNorm(a,b,c,d,e,f) :=\textrm{Determinant}(A(a,b,c,d,e,f));
```

$(a, b, c, d, e, f) - >$ LinearAlgebra:-Determinant$(A(a, b, c, d, e, f))$

```
>with(LinearAlgebra);
```

```
> FieldNorm(1, 1, 0, 0, 0, 0);
```

1

```
> FieldNorm(1, 1, 1, 0, 0, 0);
```

1

```
> g(a,b,c,d,e,f) :=[evalf(ln(abs((a
+ b*seventhroot^() + c*(seventhroot^())^(2)
+ d*(seventhroot^())^(3) + e*(seventhroot^())^(4)
+ f*(seventhroot^())^(5))^(2)))),
evalf(ln(abs((a + b*seventhroot^(2) + c*(seventhroot^(2))^(2)
+ d*(seventhroot^(2))^(3) + e*(seventhroot^(2))^(4)
+ f*(seventhroot^(2))^(5))^(2)))),
evalf(ln(abs((a + b*seventhroot^(3) + c*(seventhroot^(3))^(2)
+ d*(seventhroot^(3))^(3) + e*(seventhroot^(3))^(4)
+ f*(seventhroot^(3))^(5))^(2))))];
```

```
> lu1 := g(1, 1, 0, 0, 0, 0);
```

$(1.177725212, 0.4414486205, -1.619173834)$

```
> lu2 := g(1, 1, 1, 0, 0, 0);$
```

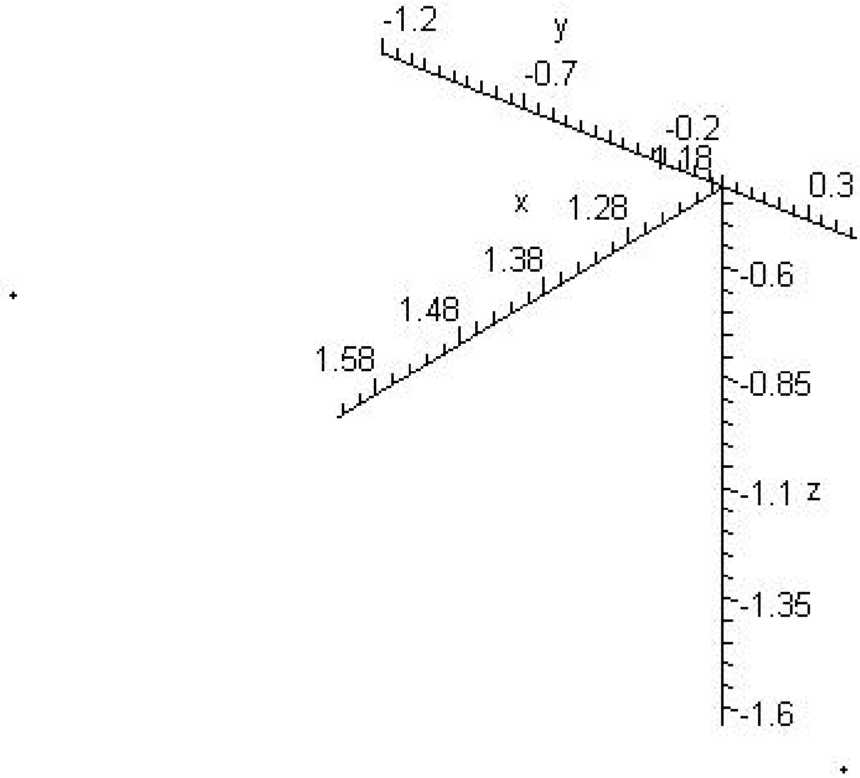
$(1.619173832, -1.177725211, -0.4414486200)$

```
> U := {lu1, lu2};
```

$\{(1.177725212, 0.4414486205, -1.619173834),$
$(1.619173832, -1.177725211, -0.4414486200)\}$

```
> with(plots);
```

```
> pointplot3d(U, axes = normal, labels = [x, y, z]);
```



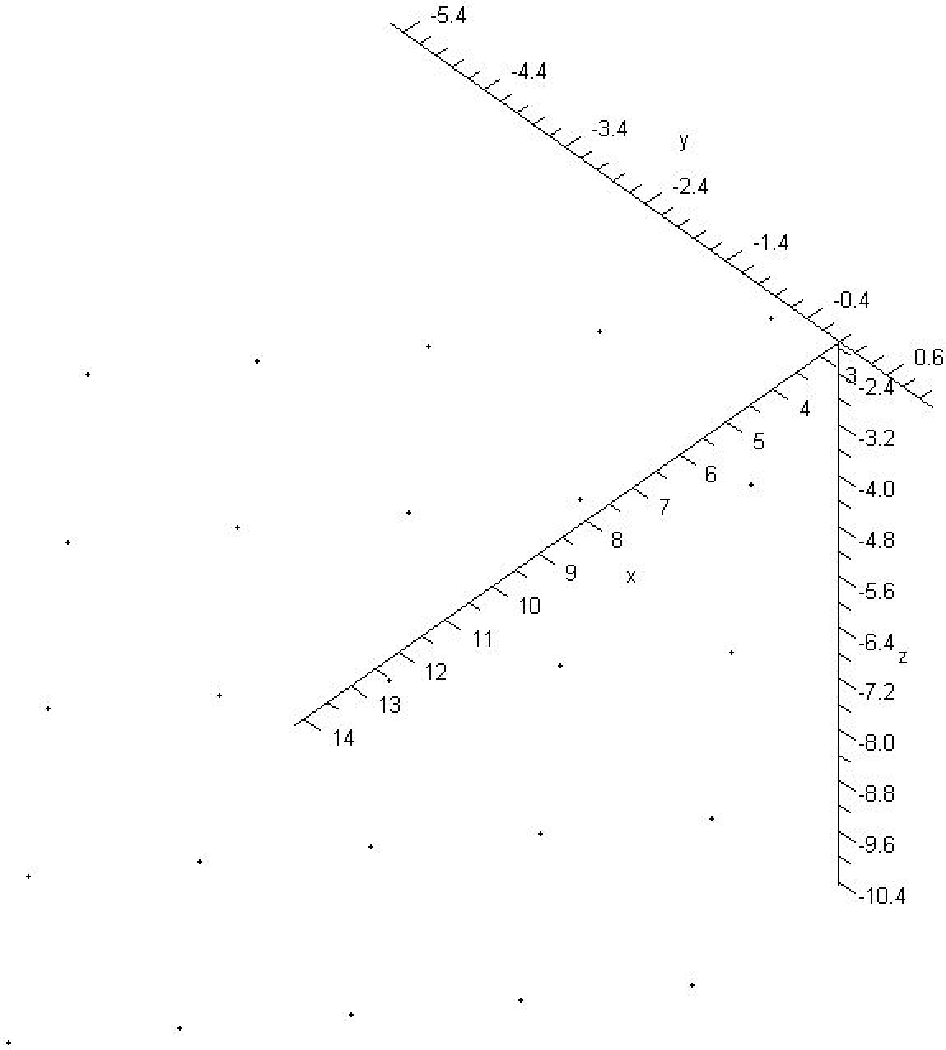

```
> points := seq(m*lu1+n*lu2, m = 1 .. 5);
```

$$\{(1.177725212, 0.4414486205, -1.619173834)$$
$$+n(1.619173832, -1.177725211, -0.4414486200),$$
$$(2.355450424, 0.8828972410, -3.238347668)$$
$$+n(1.619173832, -1.177725211, -0.4414486200),$$
$$(3.533175636, 1.324345862, -4.857521502)$$
$$+n(1.619173832, -1.177725211, -0.4414486200),$$
$$(4.710900848, 1.765794482, -6.476695336)$$
$$+n(1.619173832, -1.177725211, -0.4414486200),$$
$$(5.888626060, 2.207243102, -8.095869170)$$
$$+n(1.619173832, -1.177725211, -0.4414486200)\}$$

```
> points1 := {seq(points, n = 1 .. 5)};
```

$$\{(7.654420540, -4.269452224, -3.384968314),$$
$$(8.832145752, -3.828003603, -5.004142148),$$
$$(10.00987096, -3.386554982, -6.623315982),$$
$$(11.18759618, -2.945106362, -8.242489816),$$
$$(12.36532139, -2.503657742, -9.861663650),$$
$$(9.273594372, -5.447177434, -3.826416934),$$
$$(10.45131958, -5.005728814, -5.445590768),$$
$$(11.62904480, -4.564280193, -7.064764602),$$
$$(12.80677001, -4.122831573, -8.683938436),$$
$$(13.98449522, -3.681382953, -10.30311227),$$
$$(2.796899044, -0.7362765905, -2.060622454),$$
$$(3.974624256, -0.2948279700, -3.679796288),$$
$$(5.152349468, 0.146620651, -5.298970122),$$
$$(6.330074680, 0.588069271, -6.918143956),$$
$$(6.771523300, -1.031104560, -5.740418742),$$
$$(7.949248512, -0.589655940, -7.359592576),$$
$$(9.126973724, -0.148207320, -8.978766410),$$
$$(8.390697132, -2.208829771, -6.181867362),$$
$$(9.568422344, -1.767381151, -7.801041196),$$
$$(6.035246708, -3.091727012, -2.943519694),$$
$$(4.416072876, -1.914001802, -2.502071074),$$
$$(7.507799892, 1.029517891, -8.537317790),$$
$$(5.593798088, -1.472553181, -4.121244908),$$
$$(7.212971920, -2.650278392, -4.562693528),$$
$$(10.74614756, -1.325932531, -9.420215030)\}$$

```
> pointplot3d(points1, labels = [x, y, z], axes = normal);
```




```
> Mat := convert([lu1, lu2], Matrix);
```

$$\begin{bmatrix} 1.177725212 & 0.4414486205 & -1.619173834 \\ 1.619173832 & -1.177725211 & -0.4414486200 \end{bmatrix}$$

```
> with(LinearAlgebra);
```

```
> M1 := SubMatrix(Mat, [1, 2], [1, 2]);
```

$$\begin{bmatrix} 1.177725212 & 0.4414486205 \\ 1.619173832 & -1.177725211 \end{bmatrix}$$

```
> M2 := SubMatrix(Mat, [1, 2], [1, 3]);
```

$$\begin{bmatrix} 1.177725212 & -1.619173834 \\ 1.619173832 & -0.4414486200 \end{bmatrix}$$

```
> M3 := SubMatrix(Mat, [1, 2], [2, 3]);
```

$$\begin{bmatrix} 0.4414486205 & -1.619173834 \\ -1.177725211 & -0.4414486200 \end{bmatrix}$$

```
> abs(Determinant(M1));
```

2.101818728

```
> abs(Determinant(M2));
```

2.101818731

```
> abs(Determinant(M3));
```

2.101818729

```
> R := abs(Determinant(M3));
```

2.101818729

```
> v := evalf(R*sqrt(3));
```

3.640456828

B.2 Numerical Confirmation of the Analytic Class Number Formula for the Specific Example in Chapter 3

We can now calculate the right hand side using the value for R_K (denoted `R` in Maple code) obtained from the previous section.

```
> h := 1;
```

1

```
> w := 14;
```

14

```
> -h*R/w;
```

-0.1501299092

We now compare this to a numerical evaluation of the residue of the zero of the Dedekind zeta function, $\zeta_K(s)$ at the value $s = 0$. We will use the software GP/PARI [8] for this purpose. Since we know the order of the zero to be $r_1 + r_2 - 1 = 2$ we can numerically evaluating the function:

$$\frac{\zeta_K(s)}{s^2}$$

at values successively closer to $s = 0$. Below we show the evaluations for $s = 10^{-12}, 10^{-17}$ and 10^{-22}.

```
(15:19) gp > dznumbf = zetakinit(x^6 + x^5 + x^4 + x^3 + x^2 + x + 1)

(15:21) gp > (zetak(dznumbf,((0.1)*(10^-10))))/((0.01)*(10^-20))
%12 = -0.15012990918189141559849523 98

(15:21) gp > (zetak(dznumbf,((0.1)*(10^-15))))/((0.01)*(10^-30))
%16 = -0.1501299091778778653758993343

(15:22) gp > (zetak(dznumbf,((0.1)*(10^-20))))/((0.01)*(10^-40))
%13 = -0.15012990917645067879
```

We clearly see that a good numerical estimate for the residue is -0.1501299092, which exactly matches our other evaluation of the value.

B.3 Dedekind Zeta Values of 2

We give here the output from GP/PARI [8], complete with explanation at each step, for experimental evaluations of $\zeta_K(2)$ for various cyclotomic fields. Please note that these do not verify the relations for each algebraic number field, they are merely evaluations, but still suffice and are convincing. This part of the Appendix is designed to be read alongside chapter 5 and so is not a detailed explanation of the theory behind the process. Also, please refer to Appendix section B.3.3 for an explanation of the GP/PARI functions used.

B.3.1 An Evaluation of $\zeta_K(2)$ for the 7-th cyclotomic field

```
(13:25) gp > l2(x) = polylog(2,x,3)
(13:28) gp > seventh = cos((2*Pi)/7) + I*sin((2*Pi)/7)
%1 = 0.6234898018587335305250048840 + 0.7818314824680298087084445267*I
(13:28) gp > dz1 = l2(seventh)
%2 = 1.004653150539948718044867981
(13:28) gp > dz2 = l2(seventh^2)
%3 = 0.8264990334718855325769912159
(13:28) gp > dz3 = l2(seventh^3)
%4 = 0.3072980220409429415951663613
(13:28) gp > dz4 = l2(seventh^4)
%5 = -0.3072980220409429415951663614
(13:29) gp > dz5 = l2(seventh^5)
%6 = -0.8264990334718855325769912159
(13:29) gp > dz6 = l2(seventh^6)
%7 = -1.004653150539948718044867981
(13:29) gp > A = [dz1,dz2,dz3;dz2,dz4,dz6;dz4,dz1,dz5]
%8 =
[1.004653150539948718044867981 0.8264990334718855325769912159
0.3072980220409429415951663613]

[0.8264990334718855325769912159 -0.3072980220409429415951663614
-1.004653150539948718044867981]

[-0.3072980220409429415951663614 1.004653150539948718044867981
-0.8264990334718855325769912159]

(13:33) gp > matdet(A)
%9 = 2.315077744378181395132240794
(13:35) gp > zetak72 =  zetak(zetakinit(polcyclo(7)),2)
%10 = 1.067786228414697019446350531
(13:39) gp > discandpifact = (Pi^6)/(7^(5/2))
%12 = 7.415733875641956814028341480
(13:39) gp > rhs7 = discandpifact*matdet(A)
%13 = 17.16800045373005051578585463
(13:40) gp > zetak72/rhs7
```

```
%14 = 0.06219630709426627793974732755
(13:41) gp > lindep([%14,1])
%15 = [1029, -64]~
(13:42) gp > %14*(1029/64)
%16 = 1.000000000000000000000000001
```

B.3.2 Generalising the Procedure

First we construct a way of generating the matrix of Bloch-Wigner values of independents elements of the Bloch group of K, for K a cyclotomic field for the p-th root of unity. Here we are focusing on when $p \in \mathbb{Z}$ is prime and in particular for $p = 3, 5, 11$ and 13. The function `matrixbloch(p)` below, generates the required matrix. This determinant of this matrix does however differ from reg_K^2 by a factor of $p^{\frac{p-1}{2}}$. We are leaving this out to keep numbers small but we include it in our final expression.

```
(16:29) gp > l2(x) = polylog(2,x,3)
(16:29) gp > matrixbloch(p) = matrix((p-1)/2,(p-1)/2,j,k,
             l2(conjvec(Mod(x^k,polcyclo(p)))[2*j]))
```

These matrices for $p = 3, 5, 11$ and 13 are

```
(16:08) gp > matrixbloch(3)
%1 =
[0.6766277376064357500141350360]

(16:08) gp > matrixbloch(5)
%2 =
[0.4250778224013279193783442420 -0.9973546913984147786672835752]

[0.9973546913984147786672835751  0.4250778224013279193783442421]


(16:37) gp > matrixbloch(11)
%3 =
[-1.011155392069913374 -0.5671811537888610513 0.1969897400319769174
0.8616115304535808410 0.8936729400529264255]

[-0.8936729400529264255 -1.011155392069913374 -0.8616115304535808412
-0.5671811537888610513 -0.1969897400319769170]

[0.1969897400319769174 -0.8936729400529264255 0.5671811537888610515
```

```
-1.011155392069913374 0.8616115304535808410]

[-0.5671811537888610513 0.8616115304535808412 -0.8936729400529264255
-0.1969897400319769170 1.011155392069913374]

[-0.8616115304535808412 0.1969897400319769174 1.011155392069913374
-0.8936729400529264255 -0.5671811537888610515]

(17:21) gp > matrixbloch(13)
%4 =
[-1.012041528824899919 -0.7611405811194661500 -0.1669177194295214159
0.4864287188720071130 0.9540405707291433922 0.8363018901407091389]

[-0.8363018901407091388 -1.012041528824899919 -0.9540405707291433922
-0.7611405811194661501 -0.4864287188720071133 -0.1669177194295214165]

[-0.1669177194295214159 0.8363018901407091387 -0.4864287188720071131
1.012041528824899919 -0.7611405811194661495 0.9540405707291433928]

[-0.4864287188720071130 0.9540405707291433922 -1.012041528824899919
0.1669177194295214158 0.8363018901407091387 -0.7611405811194661500]

[-0.7611405811194661500 0.4864287188720071132 0.8363018901407091387
-0.9540405707291433925 0.1669177194295214159 1.012041528824899919]

[-0.9540405707291433924 -0.1669177194295214159 0.7611405811194661500
0.8363018901407091388 -1.012041528824899919 -0.4864287188720071132]
```

We now evaluate approximations for the right hand side of Zagier's conjecture for $p = 3, 5, 11$ and 13.

```
(16:29) gp > discfact(p) = p^((p-2)/2)
(16:30) gp > pifact(p) = Pi^(p-1)
(16:30) gp > rhs(p) = (pifact(p)*matdet(matrixbloch(p)))/(discfact(p))
(16:30) gp > rhs3 = rhs(3)
%1 = 3.855572866452448208752535395
(16:30) gp > rhs5 = rhs(5)
%2 = 10.24077807872784170996763573
(16:30) gp > rhs11 = rhs(11)
%3 = 24.34862244144364345983358724
(16:30) gp > rhs13 = rhs(13)
%4 = 23.71184784550164302180158618
```

We now give numerical approximations for $\zeta_K(2)$ for our p-th cyclotomic fields. Due to limited computing power and the high system demand to calculate these values we have reduced the precision for the $p = 11$ and $p = 13$ evaluations.

```
(16:31) gp > zeta2(p) = zetak(zetakinit(polcyclo(p)),2)
(16:31) gp > zeta23 = zeta2(3)
%5 = 1.285190955484149402917511799
(16:31) gp > zeta25 = zeta2(5)
%6 = 1.092349661730969782396547812
(16:31) gp > \p{12}
   realprecision = 19 significant digits (12 digits displayed)
(16:32) gp > allocatemem(60000000)
(16:32) gp > zeta211 = zeta2(11)
%7 = 1.03209498358
(16:33) gp > \p{9}
   realprecision = 9 significant digits
(16:34) gp > zeta213 = zeta2(13)
%8 = 1.01950072
```

We now attempt to find the rational factor that differs our evaluations of $\zeta_K(2)$ using the Bloch-Wigner function and our direct numerical approximation of $\zeta_K(2)$.

```
(16:35) gp > \p{19}
   realprecision = 19 significant digits
(16:35) gp > ratfact3 = zeta23/rhs3
%9 = 0.3333333333333333333
(16:36) gp > ratfact5 = zeta25/rhs5
%10 = 0.1066666666666666667
(16:36) gp > ratfact11 = zeta211/rhs11
%11 = 0.04238822898750499394
(16:36) gp > ratfact13 = zeta213/rhs13
%12 = 0.04299541428
(16:36) gp > lindep{[1,ratfact3]}
%13 = [1, -3]~
(16:36) gp > lindep{[1,ratfact5]}
%14 = [-8, 75]~
(16:37) gp > lindep{[1,ratfact11]}
%15 = [20480, -483153]~
(16:37) gp > lindep{[1,ratfact13]}
%16 = [120, -2791]~
```

The rational factors for $p = 3, 5$ and 11 are all convincing, since for $p = 3$ and $p = 5$ the fractions are small, and for $p = 11$ we note that

$$\frac{20480}{483153} = \frac{2^{12} \cdot 5}{3 \cdot 11^5}.$$

However, for $p = 13$ we arrive with the rational factor $\frac{120}{2791}$. Since 2791 is a prime number, we are suspicious that our approximation, using only 9 significant figures, may not have been adequate to arrive at a good prediction of the rational factor. Our suspicions are aroused because for the rational factors of p=$3, 5, 7$ and 11 the denominator of the fraction has been a multiple of a power of p. Unfortunately, we have not found the value to more significant figures due to restraints of computing power.

Results of these findings are given in section 5.5.2.

B.3.3 GP/PARI Reference

Function	Explanation
`polylog(2,x,3)`	The Bloch-Wigner function of x, $D(x)$.
`polcyclo(n)`	Generates the n-th cyclotomic field, $n \in \mathbb{Z}$.
`zetak(zetakinit(K),a)`	The Dedekind Zeta value at a of an algebraic number field K, $\zeta_K(a)$.
`lindep([a,b])`	Determines the linear dependence on the coordinates of vector `[a,b]`, used here to determine whether a and b are rationally equivalent.
`matdet(A)`	The determinant of matrix A.

See [3] for a description of more GP/PARI functions.