Boundaries in 2d Integrable Quantum Field Theory

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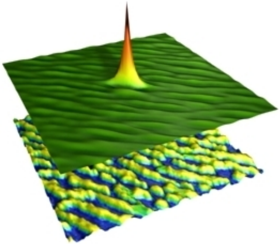

Outline

- Integrability in 2d QFT
- Adding a Boundary
- The Boundary S-matrix
- The Ising model
- Ising boundary conditions
- Correlation functions

Integrability in 2 Dimensions

Usually use Euclidean time ($y = it$) and flat coordinates given by

$$z = x + iy \qquad \bar{z} = x - iy$$
$$\partial = \partial_z = \frac{1}{2}(\partial_x - i\partial_y) \quad \bar{\partial} = \partial_{\bar{z}} = \tfrac{1}{2}(\partial_x + i\partial_y)$$

Can define a theory in this space via an action or a CFT perturbed by a relevant field. In both cases the symmetric stress tensor satisfies

$$\bar{\partial} T = \partial\Theta \quad \partial\bar{T} = \bar{\partial}\Theta$$

and Hamiltonian is given by

$$H = \int \mathrm{d}x[T + \bar{T} + 2\Theta]$$

Integrals of Motion

Assuming our model is Integrable we have an infinite set of local fields satisfying

$$\bar{\partial} T_{s+1} = \partial \Theta_{s-1} \qquad \partial \bar{T}_{s+1} = \bar{\partial} \bar{\Theta}_{s-1}$$

$$P_s = \int \mathrm{d}x (T_{s+1} + \Theta_{s-1}) \quad \bar{P}_s = \int \mathrm{d}x (\bar{T}_{s+1} + \bar{\Theta}_{s-1})$$

where $\{P_s\}$ is an infinite set of mutually commuting integrals of motion with integer spin s.
Note $H = P_1 + \bar{P}_1$

Assymptotic States

For an Integrable theory our Fock space is made up of assymptotic 'in' and 'out' states described by their rapidites $p_0 + p_1 = me^{\theta}$ and $p_0 - p_1 = me^{-\theta}$

$$| A_{a_1}(\theta_1)A_{a_2}(\theta_2)...A_{a_N}(\theta_N)\rangle = A_{a_1}(\theta_1)A_{a_2}(\theta_2)...A_{a_N}(\theta_N) \mid 0\rangle$$

where in states are ordered as $\theta_1 > \theta_2 > ... > \theta_N$ and out states as $\theta_1 < \theta_2 < ... < \theta_N$

Assymptotic states diagonalise the local Integrals of motion,

$$[P_s, A_a(\theta)] = \gamma_a^{(s)} e^{s\theta} A_a(\theta) \quad \left[\bar{P}_s, A_a(\theta)\right] = \gamma_a^{(s)} e^{-s\theta} A_a(\theta)$$

(note $\gamma_a^{(1)} = m_a$)

The S-Matrix

Scattering is purely elastic and the S-matrix is factorisable. Creation operators satisfy

$$A_{a_1}(\theta_1)A_{a_2}(\theta_2) = S^{b_1 b_2}_{a_1 a_2}(\theta_1 - \theta_2)A_{b_2}(\theta_2)A_{a_1}(\theta_1)$$

We assume **C**, **P** and **T** symmetries so $S(\theta)$ satisfies

$$S^{b_1 b_2}_{a_1 a_2}(\theta) = S^{\bar{b}_1 \bar{b}_2}_{\bar{a}_1 \bar{a}_2}(\theta) = S^{b_2 b_1}_{a_2 a_1}(\theta) = S^{\bar{a}_2 \bar{a}_1}_{\bar{b}_2 \bar{b}_1}(\theta)$$

Properties of the S-matrix

1. **The Yang-Baxter equation**

$$S_{a_1 a_2}^{c_1 c_2}(\theta) S_{c_1 a_3}^{b_1 c_3}(\theta+\theta') S_{c_2 c_3}^{b_2 b_3}(\theta') = S_{a_2 a_3}^{c_2 c_3}(\theta') S_{a_1 c_3}^{c_1 b_3}(\theta+\theta') S_{c_1 c_2}^{b_1 b_2}(\theta')$$

2. **Unitarity Condition**

$$S_{a_1 a_2}^{c_1 c_2}(\theta) S_{c_1 c_2}^{b_1 b_2}(-\theta) = \delta_{a_1}^{b_1} \delta_{a_2}^{b_2}$$

3. **Analyicity and Crossing Symmetry**

$$S_{a_1 a_2}^{b_1 b_2}(\theta) = S_{a_2 \bar{b}_1}^{b_2 \bar{a}_1}(i\pi - \theta)$$

and $S_{a_1 a_2}^{b_1 b_2}(\theta)$ is meromorphic and real at $\mathrm{Im}\theta = 0$.

Properties of the S-matrix

4. **Bootstrap Condition**
The only singularities of $S_{a_1 a_2}^{b_1 b_2}(\theta)$ in the physical strip are at $\mathrm{Re}\theta = 0$ and interpreted as bound states. These are stable so must correspond to a particle c. Position of pole in the direct channel is $iu_{a_1 a_2}^{c}$ then

$$m_{a_1}^2 + m_{a_2}^2 - m_c^2 = -2 m_{a_1} m_{a_2} \cos u_{a_1 a_2}^{c}$$

The pole term can be written as

$$S_{a_1 a_2}^{b_1 b_2}(\theta) \simeq i \frac{f_{a_1 a_2}^{c} f_{c}^{b_1 b_2}}{\theta - i u_{a_1 a_2}^{c}}$$

and the bootstrap equation is given by

$$f_{a_1 a_2}^{c} S_{c a_3}^{b b_3}(\theta) = f_{c_1 c_2}^{b} S_{a_1 c_3}^{c_1 b_3}(\theta + i \bar{u}_{a_1 \bar{c}}^{\bar{a}_2}) S_{a_2 a_3}^{c_2 b_3}(\theta + i \bar{u}_{a_2 \bar{c}}^{\bar{a}_1})$$

Adding a Boundary

Now restrict $x \in (-\infty, 0]$ and add in a boundary field with a boundary action density. In the CFT picture this can be seen a perturbed conformal boundary condition. This gives the equation

$$T_{xy}|_{x=0} = (-i)(T - \bar{T})|_{x=0} = \frac{\mathrm{d}}{\mathrm{d}y}\theta(y)$$

This condition preserves translation in the y direction and can also bee seen as a consequence of this.

An Integral of Motion

From the continuity equations

$$P_1(\mathcal{C}) = \int_{\mathcal{C}} (T\mathrm{d}z + \Theta\mathrm{d}\bar{z}) \quad \bar{P}_1(\mathcal{C}) = \int_{\mathcal{C}} (\bar{T}\mathrm{d}\bar{z} + \Theta\mathrm{d}z)$$

are independant of $\mathcal{C}$ and thus $P_1(\mathcal{C}) = \bar{P}_1(\mathcal{C}) = 0$.
Splitting the contour as follows and evaluating $P_1(\mathcal{C}_{12}) + \bar{P}_1(\mathcal{C}_{12}) = \theta(y_1) - \theta(y_2)$ allows us to say that

$$H_B(y) = \int_{-\infty}^{0} (T + \bar{T} + 2\Theta)\mathrm{d}x + \theta(y)$$

is y-independant and thus an integral of motion.

Saving Integrability

Choose boundary conditions such that

$$[T_{s+1} + \bar{\Theta}_{s-1} - \bar{T}_{s+1} - \Theta_{s-1}]|_{x=0} = \frac{\mathrm{d}}{\mathrm{d}y}\theta_s(y)$$

Then by the argument above

$$H_B^{(s)}(y) = \int_{-\infty}^{0} (T_{s+1} + \Theta_{s-1} + \bar{T}_{s+1} + \Theta_{s-1})\mathrm{d}x + \theta_s(y)$$

is also an integral of motion.

Assymptotic States in the Boundary Theory

Conserved charges act on in(out) states as

$$H_B^{(s)}|A_{a_1}(\theta_1)A_{a_2}(\theta_2)...A_{a_N}(\theta_N)\rangle_{B,in(out)} =$$
$$(\sum_{i=1}^{N} 2\gamma_{a_i}^{(s)}\cosh(s\theta_i) + h^{(s)})|A_{a_1}(\theta_1)A_{a_2}(\theta_2)...A_{a_N}(\theta_N)\rangle_{B,in(out)}$$

These conserved charges imply that the number of particles is conserved and that the out-going momenta are a permutation of minus the in-going momenta.

It is then possible to argue that the S-matrix is factoisable.

The Boundary Operator

The ground state in the boundary theory is formally written as

$$|0\rangle_B = B|0\rangle$$

The operator $B : \mathcal{H} \mapsto \mathcal{H}_B$ and satisfies

$$A_a(\theta)B = R_a^b(\theta)A_b(-\theta)B \quad [H_s, B] = h^{(s)}B$$

We can now express 'in' states in terms of 'out' states.

Correltion Functions

So far assumed boundary in space so Hilbert space is not the same as in bulk theory. Hamiltonian identified with $H_B = H_B^{(1)}$ and correlation functions computed as matrix elements

$$\langle O_1(x_1, y_1)...O_N(x_N, y_N)\rangle = \frac{{}_B\langle 0|\mathcal{T}_y(O_1(x_1, y_1)...O_N(x_N, y_N))|0\rangle_B}{{}_B\langle 0|0\rangle_B}$$

In the Euclidean picture we can take out boundary to be a boundary in time or initial state $|B\rangle$. The Hilbert space is the same as the bulk theory and correlation functions calculated via

$$\langle O_1(x_1, y_1)...O_N(x_N, y_N)\rangle = \frac{\langle 0|\mathcal{T}_x(O_1(x_1, y_1)...O_N(x_N, y_N))|B\rangle}{\langle 0|B\rangle}$$

Because of the integrals of motion in the boundary theory we must have

$$(P_s - \bar{P}_s)|B\rangle = 0$$

Boundary S-matrix Equations

1. **The Boundary Yang-Baxter Equation**

$$R_{a_2}^{c_2}(\theta_2) S_{a_1 c_2}^{c_1 d_2}(\theta_1 + \theta_2) R_{c_1}^{d_1}(\theta_1) S_{d_2 d_1}^{b_2 b_1}(\theta_1 - \theta_2) =$$
$$S_{a_1 a_2}^{c_1 c_2}(\theta_1 - \theta_2) R_{c_1}^{d_1}(\theta_1) S_{c_2 d_1}^{d_1 b_1}(\theta_1 + \theta_2) R_{d_2}^{b_2}(\theta_2)$$

2. **Boundary Unitarity**

$$R_a^c(\theta) R_c^d(-\theta) = \delta_a^b$$

Boundary S-matrix Equations

3. **Crossing Symmetry**-Need to switch pictures. $(P_s - \bar{P}_s)$ acting on assymptotic states has eigen value

$$\sum_{i=1}^{N} 2\gamma_{a_i}^{(s)} \sinh(s\theta_i)$$

So particles can only appear in the boundary state in pairs with equal mass and opposite momentum.

$$|B\rangle = \sum_{N=0}^{\infty} \int_{0<\theta_1<\ldots<\theta_N} \mathrm{d}\theta_1 \ldots \mathrm{d}\theta_N K_{2N}^{a_N \ldots a_1 b_1 \ldots b_N}(\theta_1, \ldots, \theta_N)$$
$$A_{a_N}(-\theta_N) \ldots A_{a_1}(-\theta_1) A_{b_1}(\theta_1) \ldots A_{b_N}(\theta_N)|0\rangle$$
$$= (1 + \frac{1}{2}\int_0^{\infty} \mathrm{d}\theta K^{ab}(\theta) A_a(-\theta) A_b(\theta) + \ldots)|0\rangle$$

With appropriately normalised $A_a(\theta)$ we have

$$K_{2N}^{a_N \ldots a_1 b_1 \ldots b_N}(\theta_1, \ldots, \theta_N) = R_{\bar{a}_1 \ldots \bar{a}_N}^{b_1 \ldots b_N}(\frac{i\pi}{2} - \theta_1, \ldots, \frac{i\pi}{2} - \theta_1)$$

As 'in' and 'out' states are related via the S-matrix we have the 'boundary cross-unitarity condition'

$$K^{ab}(\theta) = S_{a'b'}^{ab}(2\theta)K^{a'b'}(-\theta)$$

All K_{2N} can be expressed in terms of $K(\theta) = \frac{1}{2}K^{ab}(\theta)A_a(-\theta)A_b(\theta)$ so we can write

$$|B\rangle = \exp(\int_0^\infty \mathrm{d}\theta K(\theta))|0\rangle$$

Boundary S-matrix Equations

4. **Boundary Bootstrap Conditions**-Boundary scattering of bound state particles.

$$f_d^{ab} R_c^d(\theta) = f_c^{b_1 a_1} R_{a_1}^{a_2}(\theta + i\bar{u}_{ad}^b) S_{b_1 a_2}^{b_2 a}(2\theta + i\bar{u}_{ad}^b - i\bar{u}_{bd}^a) R_{b_2}^b(\theta - i\bar{u}_{bd}^a)$$

For 2 particles of equal mass expect a pole in $R_{\bar{a}}^b$ at $\theta = \frac{i\pi}{2} - \frac{u_{ab}^c}{2}$ ie.

$$K^{ab}(\theta) \simeq \frac{i}{2} \frac{f_c^{ab} g^c}{\theta - iu_{ab}^c}$$

Non-zero g^c indicates that $|B\rangle$ has a contribution from a zero momentum particle A_c

$$|B\rangle = (1 + g^c A_c(0) + \frac{1}{2} \int_0^\infty \mathrm{d}\theta K^{ab}(\theta) A_a(-\theta) A_b(\theta) + ...)|0\rangle$$

The Ising Model

$$\mathcal{A} = \int \mathrm{d}x\mathrm{d}y(\psi\bar{\partial}\psi - \bar{\psi}\partial\bar{\psi} + m\psi\bar{\psi})$$

Take $m > 0$ as the low-temperature phase. Ground state is degenerate $|0, \pm\rangle$ corresponding to the expectation values of the spin field $\langle\sigma(x)\rangle_{\pm} = \pm\bar{\sigma}$.
Can expand ψ and $\bar{\psi}$ as

$$\psi(x,t) = \int \mathrm{d}\theta[\omega e^{\theta/2}A(\theta)e^{imx\sinh(\theta)+imt\cosh(\theta)} + \bar{\omega}e^{\theta/2}A^{\dagger}(\theta)e^{-imx\sinh(\theta)1imt\cosh(\theta)}]$$

$$\bar{\psi}(x,t) = \int \mathrm{d}\theta[\bar{\omega}e^{-\theta/2}A(\theta)e^{imx\sinh(\theta)+imt\cosh(\theta)} + \omega e^{-\theta/2}A^{\dagger}(\theta)e^{-imx\sinh(\theta)1imt\cosh(\theta)}]$$

The S-matrix

Creation operators anti-commute:

$$A^\dagger(\theta)A^\dagger(\theta') = -A^\dagger(\theta')A^\dagger(\theta)$$

So $S = -1$ and the boundary scattering amplitude satisfies

$$A^\dagger(\theta)B = R(\theta)A^\dagger(-\theta)B \qquad R(\theta)R(-\theta) = 1$$
$$K(\theta) = -K(-\theta) \quad K(\theta) = R(\tfrac{i\pi}{2} - \theta)$$

Fixed Boundary Conditions

Removes ground state degeneracy.

$$(\psi + \bar{\psi})|_{x=0} = 0$$

In terms of creation operators

$$(\bar{\omega} e^{\theta/2} + \omega e^{-\theta/2}) A^{\dagger}(\theta) = -(\omega e^{\theta/2} + \bar{\omega} e^{-\theta/2}) A^{\dagger}(-\theta)$$

And so

$$R_{\text{fixed}}(\theta) = i \tanh\left(\frac{i\pi}{4} - \frac{\theta}{2}\right)$$

The Fixed Boundary State

In this picture fields $\chi = \omega\psi$ and $\bar{\chi} = \bar{\omega}\bar{\psi}$ have the same decomposion and creation/anihilation operators as above and boundary coditions become

$$(\chi + i\bar{\chi})|_{\tau=0}|B_{\rm fixed}\rangle = 0$$

This gives

$$|B\rangle_{\rm fixed} = \exp\{\frac{1}{2}\int_{-\infty}^{\infty} \mathrm{d}\theta K_{\rm fixed}(\theta)A^{\dagger}(-\theta)A^{\dagger}(\theta)\}|0\rangle$$

and $K_{\rm fixed}(\theta) = i\tanh(\frac{\theta}{2})$

Free Boundary Conditions

No restictions on boundary spins.

$$(\psi - \bar{\psi})_{x=0} = 0$$

Which gives

$$R_{\text{free}}(\theta) = -i\coth\left(\frac{i\pi}{4} - \frac{\theta}{2}\right) \quad \text{and} \quad K_{\text{free}}(\theta) = -i\coth\left(\frac{\theta}{2}\right)$$

The boundary state is given by

$$|B_{\text{free}}\rangle = (1 + A^{\dagger}(0))\exp\{\frac{1}{2}\int_{-\infty}^{\infty} \mathrm{d}\theta K_{\text{free}}(\theta) A^{\dagger}(-\theta) A^{\dagger}(\theta)\}|0\rangle$$

Boundary Magnetic Field

Boundary condition

$$i\frac{\mathrm{d}}{\mathrm{d}y}(\psi - \bar{\psi})_{x=0} = \frac{h^2}{2}(\psi + \bar{\psi})_{x=0}$$

and

$$R_h(\theta) = i\tanh(\frac{i\pi}{4} - \frac{\theta}{2})\frac{\kappa - i\sinh(\theta)}{\kappa + i\sinh(\theta)}$$

where $\kappa = 1 - \frac{h^2}{2m}$.

The Energy Operator

Use the Formula

$$\epsilon_0(t) = \sum_{n=0}^{\infty} \langle 0|\epsilon(x,t)|n\rangle\langle n|B\rangle$$

and the only non-zero matrix element is

$$\langle 0|\epsilon(x,t)|\beta_1\beta_2\rangle = -2\pi mi \sinh(\frac{\beta_1-\beta_2}{2})e^{-mt(\cosh(\beta_1)+\cosh(\beta_2))+imx(\sinh(\beta_1)+\mathrm{sin}}$$

From this we get

$$\epsilon_0(t,h) = -im\int_0^{\infty} \mathrm{d}\beta \sinh(\beta)\hat{K}(\beta)e^{-2mt\cosh(\beta)}$$

FIN