Goal today: Compute

$c_2(P)$, where

$$P = \begin{array}{c} S^7 \subseteq \mathbb{H}^2 \\ \big\downarrow \\ \mathbb{H}\cup\{\infty\} \cong \mathbb{HP}^1 \cong S^4 \end{array} \qquad \begin{array}{cc} (x,y) & (x,0) \\ \big\downarrow & \big\downarrow \\ xy^{-1} & \infty \\ (y\neq 0) & \end{array}$$

with right $SU(2)\cong S^3$-action

$$\omega_A = \operatorname{Im}(\bar{x}\,dx + \bar{y}\,dy)$$

$$\Longrightarrow \Omega_A = \operatorname{Im}(d\bar{x}\wedge dx + d\bar{y}\wedge dy)$$

We have been working with Ω_A

We work in a chart and trivialisation of P

$$\begin{tikzcd} i^*P_+ \arrow[r] \arrow[d, "\pi"] & S^7 =: P_+ \arrow[d] \\ \mathbb{H} \arrow[u, bend left, "s"] \arrow[r, "i"] & \mathbb{H}P^1 \\[-20pt] x \arrow[r, mapsto] & {[x:1]} \end{tikzcd}$$

$$s(x) := \frac{(x,1)}{(1+|x|^2)^{1/2}} \in S^7$$

We will compute the Chern-Weil integral with

$s^*\Omega_A$

ad-invt multilin.

<u>Claim</u>: $\Phi_k(s^*\Omega_A)$ is the unique $2k$-form

$c_{\Phi_k}(A)$ s.th.

$$\Phi_k(\Omega_A) = \underbrace{\pi^* c_{\Phi_k}(A)}_{\text{represents char. class}}$$

R_g-inv't & horiz.

Pf:

$$\pi^*\Phi_k(s^*\Omega_A) = \Phi_k((s\circ\pi)^*\Omega_A)$$

At a pt q with
$q = (s\circ\pi)(q)$
we have
$$(s\circ\pi)^*\Omega_A\big|_q = \Omega_A\big|_q .$$

and $\pi^*\Phi_k(s^*\Omega_A)$ is R_g-invariant:

$$(R_g)^*\Phi_k(\pi^* s^*\Omega_A) = \Phi_k(\underbrace{R_g^*\pi^*}_{=\pi^*} s^*\Omega_A) = \Phi_k(\pi^* s^*\Omega_A) \quad \blacksquare$$

$$s(x) = \frac{(x, 1)}{(1+|x|^2)^{1/2}}$$

$$a := s^* \omega_A$$

$$= \mathrm{Im}\left(\frac{\bar{x}}{(1+|x|^2)^{1/2}} \, d\left(\frac{x}{(1+|x|^2)^{1/2}}\right) \underbrace{+ \frac{1}{(1+|x|^2)^{1/2}} \, d\left(\frac{1}{(1+|x|^2)^{1/2}}\right)}_{\text{has zero imag. part}} \right)$$

$$= \mathrm{Im}\left(\frac{\bar{x}\,dx}{1+|x|^2} - \underbrace{\frac{\bar{x}x}{(1+|x|^2)^{1/2}} \, d\left(\frac{1}{(1+|x|^2)^{1/2}}\right)}_{\substack{\text{has zero} \\ \text{im. part} \\ \text{bec. } \bar{x}x = |x|^2}} \right)$$

$$= \mathrm{Im}\left(\frac{\bar{x}\,dx}{1+|x|^2} \right)$$

$$\Rightarrow s^*\Omega_A = s^*\left(d\omega_A + \frac{1}{2}[\omega_A \wedge \omega_A]\right)$$

$$= da + \frac{1}{2}[a \wedge a]$$

$$= \operatorname{Im}\left(\frac{d\bar{x} \wedge dx}{1+|x|^2} - \frac{1}{(1+|x|^2)^2}\, d(1+|x|^2) \wedge \bar{x}\,dx\right)$$

$$= \operatorname{Im}\left(\frac{d\bar{x} \wedge dx}{1+|x|^2} + \frac{\bar{x}dx \wedge \bar{x}dx}{(1+|x|^2)^2} - \frac{1}{(1+|x|^2)^2}(d\bar{x}\cdot x + \bar{x}dx) \wedge \bar{x}\,dx\right)$$

$=0$

$$= \operatorname{Im}\left(\frac{d\bar{x} \wedge dx}{(1+|x|^2)^2}\right)$$

$$\operatorname{Im}(d\bar{x} \wedge dx)$$

$$= \operatorname{Im}\Big[(dx_0 - i\,dx_1 - j\,dx_2 - k\,dx_3) \wedge (dx_0 + i\,dx_1 + j\,dx_2 + k\,dx_3)\Big]$$

$$= 2i\,(dx_0 \wedge dx_1 - dx_2 \wedge dx_3)$$
$$+ 2j\,(dx_0 \wedge dx_2 + dx_1 \wedge dx_3)$$
$$+ 2k\,(dx_0 \wedge dx_3 - dx_1 \wedge dx_2)$$

$$=: 2i\,\omega_i + 2j\,\omega_j + 2k\,\omega_k$$

$\omega_i, \omega_j, \omega_k$ form a ~~orthonormal~~ basis of $\Lambda^2_-(T^*\mathbb{R}^4)$ $\cong \mathbb{H}$

$$\mathbb{H} \longrightarrow u(2), \quad 1 \longmapsto \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$$

$$\cup$$

$$\operatorname{Im}\mathbb{H} \xrightarrow{\cong} su(2) \subseteq u(2) \cong \mathbb{R}^4 \quad \text{not an isom.}$$

$$\langle i, j, k \rangle_{\mathbb{R}} \qquad \left\langle \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\rangle_{\mathbb{R}}$$

$$\langle X, Y \rangle_{su(2)} := +\operatorname{tr}(XY^*) = -\operatorname{tr}(XY)$$

On $\mathbb{H}$: $\langle h, g \rangle := \operatorname{Re}(h\bar{g})$

$\|i\|_{\mathbb{H}} = 1$

$\left\|\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\right\|_{u(2)} = \sqrt{2}$

So $\operatorname{Re}_{\mathbb{H}} \mathrel{\hat{=}} \frac{1}{2}\operatorname{tr}_{u(2)}$

$$c_2(P_+) = \frac{1}{8\pi^2}\int_{S^4} \operatorname{tr}(F_A \wedge F_A)$$

$$= \frac{1}{4\pi^2}\int_{S^4} \operatorname{Re}(F_A \wedge F_A)$$

under identif. with $\mathbb{H}$

$$= \frac{1}{4\pi^2}\int_{i(\mathbb{H})} \operatorname{Re}(F_A \wedge F_A)$$

$$c_2(P_+) = \frac{1}{4\pi^2}\int_H \mathrm{Re}\,(s^*\Omega_A \wedge s^*\Omega_A)$$

$$= \frac{1}{4\pi^2}\int_H \frac{1}{(1+|x|^2)^4}\,\mathrm{Re}\big(\mathrm{Im}(d\bar{x}\wedge dx) \wedge \mathrm{Im}(d\bar{x}\wedge dx)\big)$$

$$\mathrm{Re}\,(\mathrm{Im}(d\bar{x}\wedge dx)\wedge \mathrm{Im}(d\bar{x}\wedge dx))$$
$$= -4(\omega_i\wedge\omega_i + \omega_j\wedge\omega_j + \omega_k\wedge\omega_k)$$

each $\omega_i \wedge \omega_i$
$$= -2\, dx_0\wedge dx_1\wedge dx_2\wedge dx_3$$
$$= \omega_j\wedge\omega_j$$
$$= \omega_k\wedge\omega_k$$

$$= 24\cdot dx_0\wedge dx_1\wedge dx_2\wedge dx_3$$

$$= \frac{24}{4\pi^2}\int_H \frac{1}{(1+|x|^2)^4}\, dx_0\wedge dx_1\wedge dx_2\wedge dx_3$$

Polar
$=$
coo.

$\mathrm{vol}(S^3_1)$

radius

$= 2\pi^2$

$$12 \cdot \int_0^\infty \frac{r^3}{(1+r^2)^4}\, dr$$

$$\begin{cases} t = r^2 \\ \Rightarrow dt = 2r\, dr \end{cases}$$

$$= 6 \cdot \underbrace{\int_0^\infty \frac{t}{(1+t)^4}\, dt}_{= \frac{1}{6}} \quad (*)$$

$$= 1. \qquad \square$$

$$\Rightarrow \langle c_2(P_+), [S^4] \rangle = 1$$

By the way

$(z_1, z_2) q$

$:= (\bar{q} z_1, \bar{q} z_2)$

gives

$P_- \longrightarrow S^4$

$$\begin{array}{c} S^7 =: P_- \\ \downarrow \\ \mathbb{H}P^1 \end{array}$$

Fact: The geometric conn. A_- as above is anti self-dual, and $c_2(P_-) = -1$

$$(\ast)\quad \int_0^\infty \frac{1+t-1}{(1+t)^4}\,dt$$

$$= \int_0^\infty \frac{dt}{(1+t)^3} - \int_0^1 \frac{1}{(1+t)^4}\,dt$$

$$= \left(\frac{1}{-2}(1+t)^{-2}\right)\Big|_0^\infty - \left(\frac{1}{-3}\right)(1+t)^{-3}\Big|_0^\infty$$

$$= \frac{1}{2} - \frac{1}{3}$$

$$= \frac{1}{6}$$