Exercises

1. $$S^1 \to S^{2n+1} =: P$$
$$\downarrow$$
$$\mathbb{CP}^n$$

S^1-bdle with S^1-action:

$$((z_0, \dots, z_n), w) \longmapsto (z_0 w, \dots, z_n w)$$
$$w \in S^1 \subseteq \mathbb{C}$$

$$\rho_k : \begin{cases} S^1 \longrightarrow \mathrm{Aut}(\mathbb{C}) \\ z \longmapsto \mathrm{mult}_{z^k} \end{cases}$$

$$k \in \mathbb{Z}$$

$\leadsto$ associated bundle

$$P \times_{\rho_k} \mathbb{C}$$
$$\downarrow$$
$$\mathbb{CP}^n$$

complex line bundle

(Btw.: $E = P \times_G V = P \times V / G$ ← acting by $R_g \times \rho(g^{-1})$

then $\forall p \in P$

$$V \to E_{\pi(p)}$$

$$v \mapsto [p, v]$$ ← eq. class of (p, v) wrt. G-action on $P \times V$

is an isom.)

Otoh

$$H = \{ (\ell, \underline{z}) \mid \ell \in \mathbb{CP}^n, \; \underline{z} \in \ell \}$$

$$\subseteq \mathbb{CP}^n \times \mathbb{C}^{n+1}$$

Claim:

$$S^{2n+1} \times_{S_k} \mathbb{C} \cong H^{\otimes k} \quad \text{if } k \geq 0$$

$$\cong (H^*)^{\otimes |k|} \quad \text{if } k < 0$$

where $H^* = \mathrm{Hom}_{\mathbb{C}}(H, \mathbb{C})$

Let's try to define

start with S_1

$$\begin{array}{ccc} P \times \mathbb{C} & \xrightarrow{f_1} & H \\ \downarrow & \overset{\bar{f}_1}{\nearrow} & \downarrow \\ P \times_{S_1} \mathbb{C} & \longrightarrow & \mathbb{CP}^n \end{array}$$

Suggestions for f_1 ?

$$((z_0, \dots, z_n), w) \overset{f_1}{\longmapsto} ([z_0, \dots, z_n], w \cdot (z_0, \dots, z_n))$$

$$S^{2n+1} \subseteq \mathbb{C}^{n+1}$$

for S_1 we have:

$$((z_0, \dots, z_n), w) \sim ((z_0 u, \dots, z_n u), u^{-1} w)$$

$u \in S^1$

$S_1(u^{-1}) = u^{-1}$!

$\Longrightarrow$ f_1 descends to a bundle homom.

$$S^{2n+1} \times_{S_1} \mathbb{C} \xrightarrow{\bar{f}_1} H$$

It is an isom. on fibers

$\Longrightarrow$ it is a bundle isom. because of:

General fact:

$$GL(n) \longrightarrow GL(n)$$

$$\left(\mathrm{Aut}(V) \longrightarrow \mathrm{Aut}(V)\right)$$

$$B \longmapsto B^{-1}$$

is a smooth map (polynomial for $U(n)$, $O(n)$, ...)

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applied here.

$k>0$ $\quad P \times \mathbb{C} \xrightarrow{f_k} H^{\otimes k}$

$$((z_0, \dots, z_n), w) \longmapsto$$

$=: \underline{z}$

$$([z_0 : \dots : z_n], w \cdot \underline{z} \otimes \dots \otimes \underline{z})$$

Under the action of G on $P \times \mathbb{C}$ via $R_g \times S_k$ we have

$$(\underline{z}, w) \sim (\underline{z} \cdot u, u^{-k} w)$$

$u \in S^1$ because

$$\underline{z} \cdot u \otimes \dots \otimes \underline{z} \cdot u = u^k \cdot \underline{z}^{\otimes k}$$

As before:

f_k descends to

$$Pr_{S_k}\mathbb{C} \xrightarrow{\bar{f}_k} H^{\otimes k}$$

which is a fibrewise isom. $\Rightarrow$ bundle isom.

$\overline{k<0}$:

$\underline{k=-1}$ $\quad H^* = \mathrm{Hom}(H, \mathbb{C})$

Candidate?

$$P \times \mathbb{C} \xrightarrow{f_{-1}} H^*$$

$$(\underline{z}, w) \longmapsto w \cdot \langle \underline{z}, - \rangle_{\mathbb{C}}$$

inner product on $\mathbb{C}^{n+1}$, linear in second argument (sesquilin, i.e. antilin. in first argument)

$$\underline{z}^* := \langle \underline{z}, - \rangle$$

$$(\underline{z}, w)$$

$\downarrow \cdot u$ ← S^1-action with S_{-1}

$$(\underline{z} \cdot u, u \cdot w) \xmapsto{f_{-1}} \underbrace{u \cdot w \langle \underline{z} \cdot u, - \rangle}$$

$$= u \cdot w \cdot \bar{u} \langle \underline{z}, - \rangle$$

$$= w \cdot \langle \underline{z}, \dots \rangle$$

so f_{-1} descends to

$$P \times_{S_{-1}} \mathbb{C} \xrightarrow{f_{-1}} H^* \quad \underline{\text{isom}}.$$

similarly

$$P \times_{S_k} \mathbb{C} \xrightarrow{f_k} (H^*)^{\otimes |k|}$$

$$[\underline{z}, w] \longmapsto w \cdot \underbrace{\underline{z}^* \otimes \cdots \otimes \underline{z}^*}_{|k|\text{-times}}$$

- $[X, Y]^\# = [X^\#, Y^\#]$

- $$\begin{array}{ccc} TP_p & \xleftarrow{\ \#\ } & \mathfrak{g} \\ {\scriptstyle dR_g}\downarrow & & \downarrow{\scriptstyle \mathrm{ad}_{g^{-1}}} \\ TP_{pg} & \xleftarrow{\ \#\ } & \mathfrak{g} \end{array}$$

$$(dR_g)(X^\#) = (\mathrm{ad}_{g^{-1}}(X))^\# :$$

- $$X^\#_{pg} = \frac{d}{dt}\Big|_{t=0}(pg e^{tX})$$

$$= \frac{d}{dt}\Big|_{t=0}(p\underbrace{g e^{tX} g^{-1}}_{= \mathrm{Ad}_g e^{tX}} g)$$

$$= \frac{d}{dt}\Big|_{t=0} R_g \circ R_{\mathrm{Ad}_g e^{tX}}(p)$$

$$= dR_g$$

$$\mathrm{ad}_{g^{-1}}(X) = \frac{d}{dt}\Big|_{t=0} \mathrm{Ad}_{g^{-1}}(e^{tX})$$

$$\Rightarrow \left(\mathrm{ad}_{g^{-1}}(X)\right)^{\#}_{pg}$$

$$= \frac{d}{ds}\Big|_{s=0} pg\, e^{s\,\mathrm{ad}_{g^{-1}}(X)}$$

Prop. of exp-map

$$= \frac{d}{ds}\Big|_{s=0} p\underbrace{g\, g^{-1}}_{1} e^{sX} g$$

$$= \frac{d}{ds}\Big|_{s=0} p\, e^{sX} g$$

$$= \frac{d}{ds}\Big|_{s=0} R_g(p\, e^{sX})$$

$$= dR_g\left(X^{\#}_{p}\right)$$

∎

$$[X,Y]^\# = [X^\#, Y^\#]$$

Recall:

* $[X,Y] = \frac{d}{ds}\Big|_{s=0} \mathrm{ad}_{e^{sX}}(Y)$

* Lie-bracket of v.f. on N

  If ξ, η are v.f. on a mfld

  ϕ^t_ξ flow of ξ, meaning:

  solves: $\frac{d}{dt}\phi^t_\xi(p) = \xi(\phi^t_\xi(p))$

$$[\xi,\eta](p) = \frac{d}{dt}\Big|_{t=0} d\phi^{-t}_\xi(\eta_{\phi^t_\xi(p)})$$

either def^n or theorem to be equiv.

this is a path in T_pN

Here: $G \to P$

$\downarrow$

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* $\phi^t_{X^\#} = R_{e^{tX}}$ by def^n of $X^\#$:

$$\frac{d}{dt} R_{e^{tX}}(p) = \frac{d}{ds}\Big|_{s=0} \underbrace{R_{e^{(t+s)(X)}}(p)}$$

$$= X^{\#}_{p e^{tX}} = X^{\#}_{R_{e^{tX}}(p)}$$

because

$$R_{e^{(t+s)X}} = R_{e^{sX}} \circ R_{e^{tX}}.$$

$$[X^{\#}, Y^{\#}](p)$$

$$= \frac{d}{dt}\Big|_{t=0} d R_{e^{-tX}} \left(Y^{\#}_{e^{tX}(p)} \right)$$

$$= \frac{d}{dt}\Big|_{t=0} dR_{e^{-tX}} \frac{d}{ds}\Big|_{s=0} (p e^{tX}) e^{sY}$$

$$= \frac{d}{dt}\Big|_{t=0} \frac{d}{ds}\Big|_{s=0} R_{e^{tX} e^{sY} e^{-tX}}(p)$$

$$= \frac{d}{dt}\Big|_{t=0} R_{\mathrm{ad}_{e^{tX}}(Y)}(p)$$

$$= [X, Y]^{\#}(p).$$