FORMULA SHEET for MATH 3101/41820 : FLUID MECHANICS

Some vector identities:

$$\nabla \cdot (f\boldsymbol{A}) = (\nabla f) \cdot \boldsymbol{A} + f\nabla \cdot \boldsymbol{A} \tag{1}$$

$$\nabla \times (f\boldsymbol{A}) = (\nabla f) \times \boldsymbol{A} + f\nabla \times \boldsymbol{A} \tag{2}$$

$$\nabla \times (\nabla \times \boldsymbol{A}) = \nabla(\nabla \cdot \boldsymbol{A}) - \Delta\boldsymbol{A} \tag{3}$$

$$(\boldsymbol{A} \cdot \nabla)\boldsymbol{A} = \tfrac{1}{2}\nabla|\boldsymbol{A}|^2 - \boldsymbol{A} \times (\nabla \times \boldsymbol{A}) \tag{4}$$

$$\nabla \cdot (\boldsymbol{A} \times \boldsymbol{B}) = \boldsymbol{B} \cdot (\nabla \times \boldsymbol{A}) - \boldsymbol{A} \cdot (\nabla \times \boldsymbol{B}) \tag{5}$$

$$\nabla \times (\boldsymbol{A} \times \boldsymbol{B}) = (\boldsymbol{B} \cdot \nabla)\boldsymbol{A} - (\boldsymbol{A} \cdot \nabla)\boldsymbol{B} + \boldsymbol{A}\nabla \cdot \boldsymbol{B} - \boldsymbol{B}\nabla \cdot \boldsymbol{A} \tag{6}$$

$$\nabla(\boldsymbol{A} \cdot \boldsymbol{B}) = (\boldsymbol{A} \cdot \nabla)\boldsymbol{B} + (\boldsymbol{B} \cdot \nabla)\boldsymbol{A} + \boldsymbol{A} \times (\nabla \times \boldsymbol{B}) + \boldsymbol{B} \times (\nabla \times \boldsymbol{A}) \tag{7}$$

In cylindrical coordinates (r, θ, z):

$$\nabla f = \operatorname{grad} f = \boldsymbol{e}_r\partial_r f + \frac{\boldsymbol{e}_\theta}{r}\partial_\theta f + \boldsymbol{e}_z\partial_z f \tag{8}$$

$$\nabla \cdot \boldsymbol{A} = \operatorname{div} \boldsymbol{A} = \frac{1}{r}\partial_r(rA_r) + \frac{1}{r}\partial_\theta A_\theta + \partial_z A_z \tag{9}$$

$$\nabla \times \boldsymbol{A} = \operatorname{rot} \boldsymbol{A} = \left(\frac{1}{r}\partial_\theta A_z - \partial_z A_\theta\right)\boldsymbol{e}_r + \left(\partial_z A_r - \partial_r A_z\right)\boldsymbol{e}_\theta + \frac{1}{r}\left(\partial_r(rA_\theta) - \partial_\theta A_r\right)\boldsymbol{e}_z \tag{10}$$

$$\Delta f = \frac{1}{r}\partial_r\left(r\,\partial_r f\right) + \frac{1}{r^2}\partial_{\theta\theta} f + \partial_{zz} f \tag{11}$$

$$\Delta\boldsymbol{A} = \left(\Delta A_r - \frac{1}{r^2}A_r - \frac{2}{r^2}\partial_\theta A_\theta\right)\boldsymbol{e}_r + \left(\Delta A_\theta + \frac{2}{r^2}\partial_\theta A_r - \frac{1}{r^2}A_\theta\right)\boldsymbol{e}_\theta + \boldsymbol{e}_z\Delta A_z \tag{12}$$

$$(\boldsymbol{B} \cdot \nabla)\boldsymbol{A} = \boldsymbol{e}_r\left(\boldsymbol{B} \cdot \nabla A_r - \frac{B_\theta A_\theta}{r}\right) + \boldsymbol{e}_\theta\left(\boldsymbol{B} \cdot \nabla A_\theta + \frac{B_\theta A_r}{r}\right) + \boldsymbol{e}_z\left(\boldsymbol{B} \cdot \nabla A_z\right) \tag{13}$$

In spherical coordinates (r, θ, ϕ):

$$\nabla f = \operatorname{grad} f = \boldsymbol{e}_r\partial_r f + \frac{\boldsymbol{e}_\theta}{r}\partial_\theta f + \frac{\boldsymbol{e}_\phi}{r\sin\theta}\partial_\phi f \tag{14}$$

$$\nabla \cdot \boldsymbol{A} = \operatorname{div} \boldsymbol{A} = \frac{1}{r^2}\partial_r\left(r^2 A_r\right) + \frac{1}{r\sin\theta}\partial_\theta\left(A_\theta\sin\theta\right) + \frac{1}{r\sin\theta}\partial_\phi A_\phi \tag{15}$$

$$\nabla \times \boldsymbol{A} = \operatorname{rot} \boldsymbol{A} = \frac{\boldsymbol{e}_r}{r\sin\theta}\left(\partial_\theta(A_\phi\sin\theta) - \partial_\phi A_\theta\right) + \frac{\boldsymbol{e}_\theta}{r}\left(\frac{1}{\sin\theta}\partial_\phi A_r - \partial_r(rA_\phi)\right)$$
$$+ \frac{\boldsymbol{e}_\phi}{r}\left(\partial_r(rA_\theta) - \partial_\theta A_r\right) \tag{16}$$

$$\Delta f = \frac{1}{r}\partial_{rr}(rf) + \frac{1}{r^2\sin\theta}\partial_\theta\left(\sin\theta\partial_\theta f\right) + \frac{1}{r^2\sin^2\theta}\partial_{\phi\phi} f \tag{17}$$

$$\Delta\boldsymbol{A} = \left(\Delta A_r - \frac{2}{r^2}A_r - \frac{2}{r^2\sin\theta}\left[\partial_\theta(\sin\theta A_\theta) + \partial_\phi A_\phi\right]\right)\boldsymbol{e}_r$$
$$+ \left(\Delta A_\theta + \frac{2}{r^2}\partial_\theta A_r - \frac{A_\theta}{r^2\sin^2\theta} - \frac{2\cos\theta}{r^2\sin^2\theta}\partial_\phi A_\phi\right)\boldsymbol{e}_\theta$$
$$+ \left(\Delta A_\phi + \frac{2}{r^2\sin\theta}\partial_\phi A_r + \frac{2\cos\theta}{r^2\sin^2\theta}\partial_\phi A_\theta - \frac{A_\phi}{r^2\sin^2\theta}\right)\boldsymbol{e}_\phi \tag{18}$$

$$(\boldsymbol{B} \cdot \nabla)\boldsymbol{A} = \boldsymbol{e}_r\left(\boldsymbol{B} \cdot \nabla A_r - \frac{B_\theta A_\theta}{r} - \frac{B_\phi A_\phi}{r}\right) + \boldsymbol{e}_\theta\left(\boldsymbol{B} \cdot \nabla A_\theta - \frac{B_\phi A_\phi}{r}\cot\theta + \frac{B_\theta A_r}{r}\right)$$
$$+ \boldsymbol{e}_\phi\left(\boldsymbol{B} \cdot \nabla A_\phi + \frac{B_\phi A_r}{r} + \frac{B_\phi A_\theta}{r}\cot\theta\right) \tag{19}$$

Bessel functions $u(r) = J_n(r)$ and $u(r) = Y_n(r)$ are solutions to the ODE

$$r^2 u'' + ru' + \left(r^2 - n^2\right)u = 0. \tag{20}$$

Both $J_n(r)$ and $Y_n(r) \to 0$ as $r \to \infty$; $J_n(0) = \delta_{n0}$, and $|Y_n(r)| \to \infty$ as $r \to 0$.