Durham University
Pavel Tumarkin

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Topics in Combinatorics IV, Problems Class 7 (Week 16)

7.1. List the roots of G_2 and draw the Hasse diagram of the root poset.

The Weyl group is $I_2(6)$, so it contains six reflections. We need to express six positive roots as linear combinations of simple roots α_1 and α_2. We assume that α_1 is a short root and α_2 is long.

As $\langle \alpha_2 \mid \alpha_1 \rangle = -3$ and $\langle \alpha_1 \mid \alpha_2 \rangle = -1$, we see that

$$3 = \frac{\langle \alpha_2 \mid \alpha_1 \rangle}{\langle \alpha_1 \mid \alpha_2 \rangle} = \frac{(\alpha_2, \alpha_2)}{(\alpha_1, \alpha_1)},$$

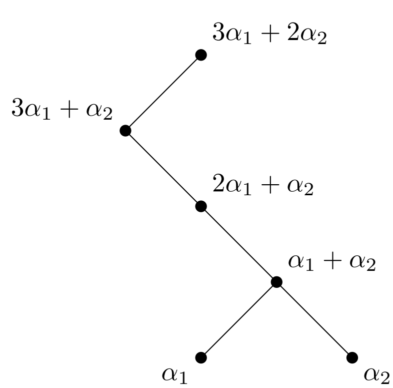
so we may assume $(\alpha_1, \alpha_1) = 1$ and $(\alpha_2, \alpha_2) = 3$. Then we have

$$(\alpha_1, \alpha_2) = \|\alpha_1\| \|\alpha_2\| (-\cos \frac{\pi}{6}) = \sqrt{3}(-\frac{\sqrt{3}}{2}) = -\frac{3}{2}$$

Due to Exercise 9.15, we have $\alpha_1 + \alpha_2 \in \Delta$, note that $\alpha_1 + \alpha_2 = r_{\alpha_2}(\alpha_1)$. Also, $r_{\alpha_1}(\alpha_2) = \alpha_2 - \langle \alpha_2 \mid \alpha_1 \rangle \alpha_1 = \alpha_2 + 3\alpha_1 \in \Delta$. As $(\alpha_1 + \alpha_2, \alpha_1) = 1 - \frac{3}{2} < 0$, we have $\alpha_2 + 2\alpha_1 \in \Delta$.

Finally, let us compute $(\alpha_2 + 3\alpha_1, \alpha_i)$. We have $(\alpha_2 + 3\alpha_1, \alpha_1) = -\frac{3}{2} + 3 > 0$, and $(\alpha_2 + 3\alpha_1, \alpha_2) = 3 + -3\frac{3}{2} < 0$, so $\alpha_2 + 3\alpha_1 + \alpha_2 = 2\alpha_2 + 3\alpha_1 \in \Delta$.

Computing lengths, we see that the roots $\alpha_1, \alpha_1 + \alpha_2$ and $2\alpha_1 + \alpha_2$ are short, and the others are long. The highest root is $2\alpha_2 + 3\alpha_1$. We get the following Hasse diagram of the root poset.



7.2. Draw the Hasse diagram of the root poset of the root system of type A_3.

The Weyl group contains 6 reflections (cf. HW 13.1), so we need to express six positive roots as linear combinations of simple roots α_1, α_2 and α_3.

We have simple roots $\alpha_1, \alpha_2, \alpha_3$, as well as $\alpha_1 + \alpha_2$ and $\alpha_2 + \alpha_3$ (due to Exercise 9.15). Now, $(\alpha_1 + \alpha_2, \alpha_3) = (\alpha_2, \alpha_3) < 0$, so $\alpha_1 + \alpha_2 + \alpha_3 \in \Delta$. Thus, we get the Hasse diagram