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Topics in Combinatorics IV

How an exam could look like

Section A

E.1. (a) (HW 1.1) Compute the number of Dyck paths of length $2n$ which start with two steps up.

(b) (HW 3.1) Denote by $p_k(n)$ the number of Young diagrams $\lambda \vdash n$ with k rows. Show that

$$p_1(n) + p_2(n) + \cdots + p_k(n) = p_k(n+k)$$

E.2. (a) (HW 7.2) Draw the Hasse diagram of the poset of order ideals of the Boolean lattice B_3 (identifying elements at every vertex). Identify join-irreducible elements of $J(B_3)$. (The latter is actually a hint.)

(b) (HW 7.3) Show that the set Π_n of set partitions of $[n]$ ordered by refinement is a lattice. Is it distributive?

E.3. (a) (HW 8.3) Let $w = 2\,6\,5\,1\,4\,8\,7\,10\,9\,3 \in S_{10}$. Apply the RSK algorithm to w to obtain SYT P and Q.

(b) (HW 8.4) Let (P, Q) be SYT of shape $\lambda = (4, 2, 2, 2) \vdash 10$, where

$$P = \begin{array}{|c|c|c|c|} \hline 1 & 3 & 4 & 10 \\ \hline 2 & 5 \\ \cline{1-2} 6 & 7 \\ \cline{1-2} 8 & 9 \\ \cline{1-2} \end{array} \qquad Q = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 5 & 6 \\ \hline 3 & 4 \\ \cline{1-2} 7 & 8 \\ \cline{1-2} 9 & 10 \\ \cline{1-2} \end{array}$$

Construct $w \in S_{10}$ which is taken to the pair (P, Q) by the RSK algorithm.

E.4. (a) (HW 16.2) Let Δ be a root system. Show that the highest root $\tilde{\alpha}_0$ is always long, i.e. $(\tilde{\alpha}_0, \tilde{\alpha}_0) \geq (\alpha, \alpha)$ for any $\alpha \in \Delta$.

(b) (HW 19.1(a)) Compute the Coxeter number and exponents of Weyl group of type C_4.

Section B

E.5. (HW 2.4) We say that a Dyck path has a *hill* at point $2i+1$ if it passes through points $(2i, 0)$ and $(2i+2, 0)$. Denote by F_k the number of *hill-free* Dyck paths of length $2k$, i.e. Dyck paths without hills.

(a) Compute F_k for $k \leq 5$.

(b) Show that numbers F_k satisfy the following equation:

$$C_n = F_n + \sum_{k=0}^{n-1} F_k C_{n-k-1},$$

where C_k are Catalan numbers.
Hint: consider the first hill from the left.

(c) Compute the generating function $F(x)$ of the sequence (F_k). Show that

$$F(x) = \frac{1}{1 - x^2 C(x)^2},$$

where $C(x)$ is the generating function for Catalan numbers.

E.6. (a) (HW 8.1) Show that the poset $J(P)$ of order ideals of a poset P is a distributive lattice.

(b) (HW 8.2) Given a poset P with $|P| = n$, construct a map from the set of linear extensions of P to the set of saturated chains of $J(P)$ by taking $\varphi : P \to [n]$ to the chain $\hat{0} = \emptyset \lessdot I_1 \lessdot I_2 \lessdot \ldots \lessdot I_n = \hat{1}$, where $I_j = \varphi^{-1}([j])$. Show that this map is a bijection.

E.7. (HW 16.1) Let Δ be a root system, $\Pi = \{\alpha_i\}$ is a set of simple roots.

(a) Show that $r_{\alpha_i}(\Delta^+ \setminus \alpha_i) = \Delta^+ \setminus \alpha_i$. In other words, r_{α_i} takes all positive roots except α to positive roots.

(b) Let $w \in W$, $\alpha \in \Pi$. Denote $n(w) = \#\{\beta \in \Delta^+ \mid w\beta \in \Delta^-\}$, i.e. the number of positive roots taken by w to negative ones. Show that if $w\alpha \in \Delta^+$ then $n(wr_\alpha) = n(w)+1$, and if $w\alpha \in \Delta^-$ then $n(wr_\alpha) = n(w) - 1$. In particular, $n(w) \leq l(w)$.

(c) Let $s_1 \ldots s_k$ be a reduced expression for w, where $s_i = r_{\alpha_i}$ are simple reflections. Show that if $n(w) < l(w)$ then there exist $i < j$ such that $s_i(s_{i+1} \ldots s_{j-1})\alpha_j = \alpha_i$.

(d) Show that $n(w) = l(w)$ for every $w \in W$.

E.8. (a) (HW 15.4) Let (G, S) be Coxeter system, $r \in R$ and $g \in G$. Show that if $r \in R(g)$ then $l(rg) < l(g)$.

(b) (HW 14.3) Let G be a group with a finite generating set S consisting of involutions, and let $\{P_s\}_{s \in S}$ be a family of subsets of G satisfying the following properties:

(1) $e \in P_s$ for every $s \in S$;

(2) $P_s \cap sP_s = \emptyset$ for every $s \in S$;

(3) For every $s, t \in S$ and $g \in G$ such that $g \in P_s$ and $gt \notin P_s$, one has $sg = gt$.

Show that $P_s = \{g \in G \mid l(sg) > l(g)\}$, and (G, S) satisfies Exchange Condition (and thus is a Coxeter system).