Durham University
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# Topics in Combinatorics IV, Homework 15 (Week 15)

**Due date** for starred problems: **Friday, February 23, 6pm**.

**15.1.** Let $\Gamma = \langle s_1, s_2, s_3 \mid s_i^2, (s_1s_2)^3, (s_2s_3)^3, (s_1s_3)^3, s_3^7 \rangle$. Show that the subgroup generated by $s_1$ and $s_2$ is trivial, although the group $\Gamma' = \langle s_1, s_2 \mid \text{all relations not containing } s_3 \rangle$ is not.

**15.2.** Let $(G, S)$ be a Coxeter system, and let $T \subset S$. Define $G_T$ to be the subgroup of $G$ generated by elements of $T$ ($G_T$ is called a *standard parabolic subgroup* of $G$).

(a) Let $w = s_1 \dots s_k$ be a word, all $s_i \in T$. Show that for any $M$-reduction $w \to w_0$ all words obtained during the procedure belong to $G_T$.

(b) Let $\Gamma = \langle T \mid s_i^2, (s_is_j)^{m_{ij}} \rangle$. Define a homomorphism $\varphi : \Gamma \to G$ by $\varphi(s_i) = s_i$. Show that $\ker \varphi$ is trivial.

(c) Show that $(G_T, T)$ is a Coxeter system.

**15.3.** ($\star$) Let $(G, S)$ be a Coxeter system, $s, t \in S$, and $m_{st} = \infty$ (i.e., there is no relation on $st$). Let $w$ be a reduced word. Show that either $s \notin r(w)$ or $t \notin r(w)$.

**15.4.** Let $(G, S)$ be Coxeter system, $r \in R$ and $g \in G$. Show that if $r \in R(g)$ then $l(rg) < l(g)$.

**15.5.** ($\star$) Let $(G, S)$ be Coxeter system such that its Coxeter diagram contains a cycle. Find an element of infinite order in $G$.