Durham University
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Epiphany 2024

Topics in Combinatorics IV, Homework 13 (Week 13)

Due date for starred problems: **Friday, February 9, 6pm**.

13.1. (a) Let G be an irreducible simply-laced finite reflection group (i.e., all m_{ij} are equal to 2 or 3), or one of H_3 and H_4. Show that all reflections of G are conjugated in G.

(b) Let G be a finite reflection group, and let $r_1, r_2 \in G$ be two reflections. Show that the dihedral subgroup generated by r_1 and r_2 is conjugated in G to a subgroup generated by some simple reflections s_i and s_j.

13.2. (a) Let G be a Coxeter group defined by $G = \langle t_1, t_2, t_3 \mid t_i^2, (t_1t_2)^2, (t_1t_3)^2, (t_2t_3)^3 \rangle$. Show that G is isomorphic to $I_2(6)$.

(b) Show that for any odd k a Coxeter group $G = \langle t_1, t_2, t_3, \mid t_i^2, (t_1t_2)^2, (t_1t_3)^2, (t_2t_3)^k \rangle$ is isomorphic to $I_2(2k)$.

13.3. ($\star$) Let $\{e_1, e_2, e_3\}$ be a standard orthonormal basis of $\mathbb{R}^3$. Let G be the group generated by reflections in all vectors of the form $e_i \pm e_j$, $i < j$.

(a) Show that G does not contain any other reflections.

(b) Find a triple of reflections $\{s_1, s_2, s_3\}$ of G such that the angles formed by the outward normals $\{\alpha_1, \alpha_2, \alpha_3\}$ to the mirrors of s_i are all non-acute (i.e., $\pi/2$ or larger).
Note: there are *many* such triples.

(c) Write down the *Gram matrix* of $\{\alpha_1, \alpha_2, \alpha_3\}$, i.e. the matrix (a_{ij}) with $a_{ij} = (\alpha_i, \alpha_j)$. Draw the corresponding Coxeter diagram.

(d) Write down the presentation of G as a Coxeter group. Find the order of G.