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Topics in Combinatorics IV, Problems Class 3 (Week 6)

The class was mostly devoted to the second assignment questions, in particular to the solutions alternative to the ones published (see solutions uploaded).

HW 4.2 We want to construct a bijection between Dyck paths of length $2n$ with k peaks and non-crossing partitions of $[n]$ with k blocks.

Take a Dyck path, and index all steps up in turn. Now for each step up find the corresponding step down and assign to it the same number. To do this observe that to a Dyck path we can associate a sequence of n opening and n closing brackets (up – opening, down – closing), such that the number of opening brackets at every moment is not less the number of closing ones; then every opening bracket has the corresponding closing one, this will correspond precisely to the matching down step).

Now every block of the partition is the indices of consecutive steps down. Since the number of peaks is equal to k, the number of blocks is also equal to k.

The easiest way to prove that the partition is non-crossing is by induction. If the Dyck path reaches the x-axis at $2m < 2n$, then it is a union of two smaller Dyck paths, and by the induction assumption our partition is a union of two non-crossing partitions of $[m]$ and $\{m+1, \dots, n\}$. If the Dyck path is always above x-axis, then observe that 1 is in the same block with n, and by removing 1 we get a shorter Dyck path which, by the induction assumption, corresponds to a non-crossing partition of $\{2, \dots, n\}$. We need to prove that if we add 1 to the block containing n, the partition will still be non-crossing. Indeed, if it is not, then there are arc $(1, j)$ and (i, m) for some $1 < i < j < m < n$ in the corresponding arc diagram. But note that j is in the same block with n, and thus there is a sequence of arcs $(j = j_0, j_1), (j_1, j_2) \dots (j_s, j_{s+1} = n)$. Then at least one of this arcs intersects the arc (i, m), which contradicts the fact the partition of $\{2, \dots, n\}$ was non-crossing.

It is easy to see that different Dyck paths give rise to different partitions, so the map is injective. Since the total number of paths and total number of partitions are the same (i.e., C_n), the map is also surjective, and thus a bijection.

3.1. Recall that given $w = w_1 w_2 \dots w_n \in S_n$, $\operatorname{inv}(w)$ is the number of inversions (i.e. pairs $i < j$ such that $w_i > w_j$), $\operatorname{des}(w)$ is the number of descents (i.e. places $i \in [n-1]$ such that $w_i > w_{i+1}$), and $\operatorname{maj}(w)$ is the sum of all $i \in [n-1]$ such that i is a descent of w.

Show that two statistics maj and inv are equidistributed.

Hint: use induction.

For $n = 1, 2$ the statement is clear. We now proceed as in the proof of Thm. 3.6. There is an $n : 1$ map $\varphi : S_n \to S_{n-1}$ which erases n in 1-line notation. We want to show that given $w \in S_{n-1}$, for every $i \in [n]$ there is an element $w^{(i)} \in S_n$ of the preimage of w under φ such that $\mathrm{maj}\,(w^{(i)}) = \mathrm{maj}\,(w) + i$. Then the results follows by induction.

Index descents of w from right to left. Note that if n in $w' \in \varphi^{-1}(w)$ is located immediately to the right of the descent number k of w, then the difference $\mathrm{maj}\,(w') - \mathrm{maj}\,(w)$ is equal to k: the k-th descent is substituted by n which is now on the place of k-th descent plus 1, and the first $k-1$ descents are shifted by one place. Further, observe that if n in $w' \in \varphi^{-1}(w)$ is located at the n-th place, then $\mathrm{maj}\,(w') = \mathrm{maj}\,(w)$. Therefore, we identified locations of n in $w' \in \varphi^{-1}(w)$ with $\mathrm{maj}\,(w') - \mathrm{maj}\,(w) = 0, \ldots, \mathrm{des}\,(w)$.

Call the entries of w that are not descents *ascents*, and index them from left to right; we exclude the last, $(n-1)$-st entry, so there are $n - 2 - \mathrm{des}\,(w)$ ascents. If n in $w' \in \varphi^{-1}(w)$ is located at the very first place, then $\mathrm{maj}\,(w') - \mathrm{maj}\,(w) = \mathrm{des}\,(w) + 1$: every descent of w is shifted by one to the right, and there is additional one on the first place. Furthermore, if n in $w' \in \varphi^{-1}(w)$ is located immediately to the right of the ascent number k, then $\mathrm{maj}\,(w') - \mathrm{maj}\,(w)$ is equal to $\mathrm{des}\,(w) + 1 + k$: if the place right after k-th ascent has number m, then there are $\mathrm{des}\,(w) - ((m-1) - k)$ descents to the right, so the difference $\mathrm{maj}\,(w') - \mathrm{maj}\,(w)$ is equal to $\mathrm{des}\,(w) - ((m-1) - k) + m = \mathrm{des}\,(w) + k + 1$. As the number of ascents is equal to $n - 2 - \mathrm{des}\,(w)$, $\mathrm{maj}\,(w') - \mathrm{maj}\,(w)$ attains all values between $\mathrm{des}\,(w) + 1$ and $\mathrm{des}\,(w) + 1 + (n - 2 - \mathrm{des}\,(w)) = n - 1$.

Therefore, as w' goes through the whole preimage of w, the difference $\mathrm{maj}\,(w') - \mathrm{maj}\,(w)$ goes through all non-negative integers from 0 to $n - 1$. The rest of the proof is identical to the one of Theorem 3.6.