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Topics in Combinatorics IV, Homework 9 (Week 9)

9.1. Denote by $r_x(P)$ an insertion of x in a partial tableau P in the RSK algorithm. Suppose that during $r_x(P)$ the elements $x_1, \dots, x_k$ are pushed down from rows $1, 2, \dots, k$ and columns $j_1, j_2, \dots, j_k$ respectively. Then

(a) $x < x_1 < \dots < x_k$;

(b) $j_1 \geq \dots \geq j_k$;

(c) if $P' = r_x(P)$, then $P'_{i,j} \leq P_{i,j}$ for all i, j.

9.2. (a) Show that

$$\sum_{\lambda \vdash n} f_\lambda = \#\{w \in S_n \mid w^2 = 1\}$$

(b) Show that

$$\sum_{\lambda \vdash n} f_\lambda = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} \frac{(2k)!}{2^k\, k!}$$

9.3. (A bit of linear algebra) Let A be a real symmetric $n \times n$ matrix with all off-diagonal elements being non-positive. Assume also that A is *indecomposable*, i.e. it cannot be made block-diagonal by any simultaneous permutation of rows and columns. Show that A is positive definite if and only if there exists a vector $v \in \mathbb{R}^n$ with all positive coordinates such that all coordinates of Av are also positive.

Hint: use *Perron-Frobenius Theorem* which states that if all entries of a square matrix are non-negative, then it has a simple positive eigenvalue μ such that μ has maximal modulus amongst all eigenvalues of A, and all the coordinates of the corresponding eigenvector are positive.