Durham University
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Topics in Combinatorics IV, Homework 6 (Week 6)

Due date for starred problems: **Friday, November 17, 6pm**.

6.1. Recall that given $w \in S_n$, $\operatorname{exc}(w)$ is the number of excedances of w (i.e. places $i \in [n]$ such that $i < w_i$).

Complete the proof of Theorem 3.13: show that statistics des and exc are equidistributed.

6.2. Let $w = w_1 w_2 \dots w_n \in S_n$, $n \geq 2$. $i \in [n]$ is a *weak excedance* of w if $w_i \geq i$. Denote by $\operatorname{wexc}(w)$ the number of weak excedances of $w \in S_n$.

Show that statistics exc and wexc $- 1$ are equidistributed.

6.3. ($\star$) Define *Eulerian numbers* $A(n,k)$ as the numbers of permutations $w \in S_n$ with $\operatorname{des}(w) = k-1$, $k \leq n$.

Show that $A(n, k+1) = (n-k)A(n-1,k) + (k+1)A(n-1,k+1)$.

6.4. ($\star$) Let P_1, P_2 be posets. A map $f : P_1 \to P_2$ is called *order-preserving* if for any $a, b \in P_1$ the relation $a \leq_{P_1} b$ implies $f(a) \leq_{P_2} f(b)$.

(a) Let P be a finite poset, and let $f : P \to P$ be an order-preserving bijection. Show that f^{-1} is also order-preserving.

(b) Show that for infinite posets the statement of part (a) may not hold.