Durham University
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Michaelmas 2023

Topics in Combinatorics IV, Homework 2 (Week 2)

Due date for starred problems: **Friday, October 20, 6pm**.

2.1. Let n be a positive integer, and let p be a prime.

(a) Show that the number of sequences of integers $n_1, \dots, n_p$, where $1 \le n_i \le n$ and at least two n_i's are distinct, is equal to $n^p - n$.

(b) Show that all cyclic shifts of any sequence from (a) are distinct.

(c) Deduce that $n^p - n$ is divisible by p.

2.2. Consider a Drunkard's walk in the segment $[0, n]$, i.e.:

- the walk starts at interger $x = i$, $0 \le i \le n$;
- the probability of steps left and right is equal to $1/2$;
- the walk ends when the drunkard reaches either $x = 0$ or $x = n$.

Denote by p_i the probability the walk starting at $x = i$ ends at point $x = n$.

(a) Show that $p_i = \frac{1}{2}p_{i-1} + \frac{1}{2}p_{i+1}$ for every $i = 1, \dots, n-1$.

(b) Compute p_i for every i.
Hint: you may need to recall some linear algebra.

(c) Deduce from (b) the result of Example 1.15 (Drunkard's walk) from lectures.

2.3. Find a bijection between the set of non-decreasing sequences $1 \le a_1 \le \dots \le a_n$ such that $a_i \le i$ and lattice paths in the $n \times n$ square from $(n, 0)$ to $(0, n)$ lying above the main diagonal (and thus, show that the number of such sequences is C_n).

2.4. ($\star$) We say that a Dyck path has a *hill* at point $2i + 1$ if it passes through points $(2i, 0)$ and $(2i + 2, 0)$. Denote by F_k the number of *hill-free* Dyck paths of length $2k$, i.e. Dyck paths without hills.

(a) Compute F_k for $k \le 5$.

(b) Show that numbers F_k satisfy the following equation:

$$C_n = F_n + \sum_{k=0}^{n-1} F_k C_{n-k-1},$$

where C_k are Catalan numbers.
Hint: consider the first hill from the left.

(c) Compute the generating function $F(x)$ of the sequence (F_k). Show that

$$F(x) = \frac{1}{1 - x^2 C(x)^2},$$

where $C(x)$ is the generating function for Catalan numbers.