Durham University   Michaelmas 2023
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Topics in Combinatorics IV, Homework 1 (Week 1)

Due date for starred problems: Friday, October 20, 6pm.

1.1. $(\star)$ Compute the number of Dyck paths of length $2n$ which start with two steps up.

1.2. $(\star)$

    (a) Let P be a Dyck path of length $2n$, let $a_1, \ldots, a_n$ be the positions of the steps "down", $1 \leq a_i \leq 2n$. Show that $a_i \geq 2i$ for every $i \leq n$.

    (b) Show that the number of strictly increasing sequences $(a_1, \ldots, a_n)$ of integers satisfying $2i \leq a_i \leq 2n$ is the n-th Catalan number C_n.

    (c) Show that the number of strictly increasing sequences $(a_1, \ldots, a_{n-1})$ of integers satisfying $1 \leq a_i \leq 2i$ is the n-th Catalan number C_n.

1.3. Show explicitly that the number of triangulations of an $(n + 2)$-gon satisfies the Catalan recursion (see Lemma 1.10 from lectures).

1.4. Find a bijection between ballot sequences of length $2n$ and bracketings of $n + 1$ variables. *Hint*: assign to every $+1$ in the sequence an opening bracket.

1.5. Given a ballot sequence $\varepsilon_1, \ldots, \varepsilon_{2n}$, one can write a sequence of differences $a_i = \varepsilon_{i+1} - \varepsilon_i$, $1 \leq i \leq 2n - 1$. Characterize all such sequences (and thus, get another definition of Catalan numbers).