Durham University
Pavel Tumarkin Epiphany 2025

Topics in Combinatorics IV, Term 2

7 Reflection groups

7.1 Linear reflections and reflection groups

We consider $\mathbb{R}^n$ with standard Euclidean scalar product $(\cdot,\cdot)$ (i.e., dot product).

Definition 7.1. Let $\alpha \in \mathbb{R}^n$. A *reflection* with respect to α is a linear map $r_\alpha : \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$r_\alpha(v) = v - \frac{2(v,\alpha)}{(\alpha,\alpha)}\alpha$$

for $v \in \mathbb{R}^n$.

It is easy to see that r_α is characterized by the following two properties: it takes α to $-\alpha$ and preserves pointwise the orthogonal complement $\alpha^\perp$.

Exercise 7.2. $r_\alpha \in O_n(\mathbb{R})$, i.e. $(r_\alpha(u), r_\alpha(v)) = (u, v)$; also, r_α is an involution, i.e. $r_\alpha^2 = \mathrm{id}$.

Definition 7.3. A *reflection group* is a group generated by reflections.

Remark. Usually "reflection group" means a *discrete reflection group*, which requires some additional geometrical properties to hold (namely, the orbit of any point should not have limit points). We will mainly be interested in *finite* reflection groups, and for these there are no extra requirements.

Example 7.4.

- Consider vectors $\alpha = (1, 0)$ and $\beta = (\cos\frac{(m-1)\pi}{m}, \sin\frac{(m-1)\pi}{m})$. Then $r_\alpha r_\beta$ is a rotation by $2\pi/m$, and the group generated by r_α and r_β is a dihedral group of order $2m$ (denoted by $I_2(m)$).
- The symmetric group S_{n+1} acts on $\mathbb{R}^{n+1}$ by permutation of coordinates, preserving the hyperplane $V_0 = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid \sum x_i = 0\}$. Every transposition (ij) is a reflection in a plane $x_i - x_j = 0$, i.e. with respect to the vector $e_i - e_j$. As S_{n+1} is generated by transpositions, it is a reflection group in $V_0 = \mathbb{R}^n$.

  One can also note that the action of S_{n+1} by permutation of coordinates of $\mathbb{R}^{n+1}$ preserves the affine hyperplane $V_1 = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid \sum x_i = 1\}$, and it also preserves the positive orthant $C_+ = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_i \geq 0\ \forall i \in [n+1]\}$. Thus, S_{n+1} preserves the *regular n-dimensional simplex* $V_1 \cap C_+$, acting on it by permutations of its vertices.

Definition 7.5. For reflection r_α, $\alpha \in \mathbb{R}^n$, the orthogonal complement $\alpha^\perp$ is called the *mirror* of r_α.

Lemma 7.6. *Let $g \in O_n(\mathbb{R})$, $\alpha \in \mathbb{R}^n$. Then $gr_\alpha g^{-1} = r_{g\alpha}$.*

Proof. We need to prove that $gr_\alpha g^{-1}$ fixes every point of $\langle g\alpha\rangle^\perp$ and takes $g\alpha$ to $-g\alpha$.

Let $(v, g\alpha) = 0$. Since $g \in O_n(\mathbb{R})$, this implies that $(g^{-1}v, \alpha) = 0$. Then

$$gr_\alpha g^{-1}(v) = g(r_\alpha(g^{-1}(v)) = g((g^{-1}(v)) = v,$$

so $gr_\alpha g^{-1}$ preserves $\langle g\alpha\rangle^\perp$ pointwise.

Also, $gr_\alpha g^{-1}(g\alpha) = gr_\alpha(\alpha) = g(-\alpha) = -g\alpha$, as required. □

In general, what can we say about *finite* reflection groups in $\mathbb{R}^n$?

First, since every reflection is orthogonal, any reflection group is a subgroup of $O_n(\mathbb{R})$. Given a finite reflection group G in $\mathbb{R}^n$, Lemma 7.6 implies that the set of mirrors of reflections of G is invariant under the action of G (i.e., G permutes its mirrors).

The set of mirrors of G decomposes $\mathbb{R}^n$ into polyhedral cones – we call them *chambers*, and the mirrors bounding a chamber are called its *walls*.

Remark. Note that, due to the invariance of the set of mirrors under G, any two chambers sharing a wall are taken to each other by the reflection in the common wall. Indeed, if we take chambers C_1 and C_2 sharing a wall $\alpha^\perp$, we can consider $C' = r_\alpha C_1$. If C' is not a chamber, then there exists a mirror $\beta^\perp$ of reflection in G intersecting C'. Applying r_α to $\beta^\perp$, we see that the image intersects C_1, which contradicts C_1 being a chamber. Now, both C' and C_2 are chambers, and they clearly have a non-empty intersection, so they must coincide.

Recall that an action of a group on a set is *transitive* if the set consists of one orbit.

Theorem 7.7. *Let G be a finite reflection group in $\mathbb{R}^n$. Consider all mirrors of reflections of G, and take any connected component of the complement, call this chamber C_0. Denote $r_{\alpha_1}, \dots, r_{\alpha_k}$ the reflections with respect to the walls of C_0. Then*

(1) G is generated by $r_{\alpha_1}, \dots, r_{\alpha_k}$.

(2) G acts transitively on the set of chambers.

(3) The dihedral angles between walls of C_0 are of the type π/m_{ij}, $m_{ij} \in \mathbb{N}_{\geq 2}$.

(4) If $g \in G$ and $gC_0 = C_0$ then $g = \mathrm{id}$.

(5) G has presentation $G = \langle r_{\alpha_1}, \dots, r_{\alpha_k} \mid r_{\alpha_i}^2, (r_{\alpha_i} r_{\alpha_j})^{m_{ij}} \rangle$ (i.e. any relation on the generators follows from these two types of relations).

Proof. Denote r_{α_i} by s_i. Take any chamber C, connect it to C_0 by a path (which does not pass through an intersection of three or more chambers). Write down the sequence of chambers intersected by the path: $C_0, C_1, \dots, C_m = C$ (chambers may repeat in the sequence). Note that any two neighboring chambers in the sequence share a wall.

Since C_1 and C_0 share a wall (say, mirror of s_{i_1}), we can write $C_1 = s_{i_1} C_0$. By Lemma 7.6, walls of C_1 are precisely mirrors of reflections $s_{i_1} s_j s_{i_1}$, $j = 1, \dots, m$. Since C_2 and C_1 share a wall (say, mirror of $s_{i_1} s_{i_2} s_{i_1}$), we can write

$$C_2 = s_{i_1} s_{i_2} s_{i_1} C_1 = s_{i_1} s_{i_2} s_{i_1} (s_{i_1} C_0) = s_{i_1} s_{i_2} C_0.$$

Continuing along the path, we see that $C = C_m = s_{i_1} s_{i_2} \dots s_{i_m} C_0$, where s_{i_j} are reflections in the walls of C_0. This proves (2).

Moreover, we have proved that any reflection in G is conjugated to at least one of s_i, which proves (1).

Now, let us prove (3). Take any s_i and s_j and consider the group generated by them. If the angle is not $p\pi/m$, then the order of $s_i s_j$ is infinite, which contradicts finiteness of G. Further, if $p \neq 1$ (we may assume that p and m are coprime), then the walls are separated by another mirror (this is actually a question about dihedral groups), which implies that there exists a mirror of G intersecting the interior of C_0 in contradiction with its definition, so (3) is also proved.

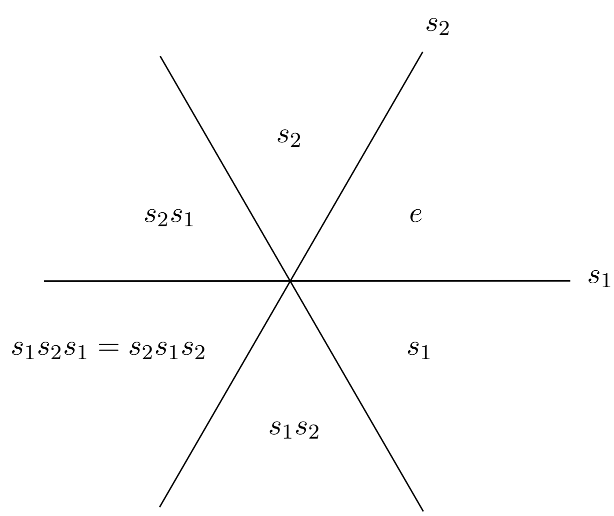
To prove (4) and (5) consider any word $s_{i_1} s_{i_2} \dots s_{i_m}$ realizing a path from C_0 to C_0 going through chambers $s_{i_1} s_{i_2} \dots s_{i_k} C_0$ for $k = 1, \dots, m-1$. Note that the relations in (5) do hold (as they hold in the corresponding dihedral groups). Moreover, they imply the same relations for reflections in walls of any chamber (due to Lemma 7.6). Further, using the relations we can contract the path to an empty one. More precisely, if a path intersects any single wall twice in a row, then applying the relation r^2 for r being the reflection in the corresponding wall we shorten the path; to go around an intersection of more than two chambers (which is a vector space of codimension two), one can use the relations (5) to substitute a subword of type $s_i s_j s_i \dots$ of length l by a word of type $s_j s_i s_j \dots$ of length $2m_{ij} - l$. Therefore, every path from C_0 to C_0 corresponds to a trivial element of G, and the word can be reduced to e by the required relations, which proves both (4) and (5).

□

Corollary 7.8. *Chambers of G are indexed by elements of G.*

Indeed, we can choose an initial chamber C_0 and then associate with any chamber $C = gC_0$ the corresponding element g.

Example. Consider $I_2(3) = S_3$, it has presentation $\langle s_1, s_2 \mid s_1^2, s_2^2, (s_1 s_2)^3 \rangle$.



Definition 7.9. Let a group G act on an open connected set X. An open connected domain $C \subset X$ is called a *fundamental domain* of the action if the following conditions are satisfied:

- $X = \bigcup_{g \in G} \overline{gC}$, where $\overline{gC}$ denotes the closure of gC;
- for any nontrivial $g \in G$, $C \cap gC = \emptyset$;
- there are finitely many $g \in G$ such that $\overline{C} \cap \overline{gC} \neq \emptyset$.

Corollary 7.10. *Any chamber C of a finite reflection group G is a fundamental domain of the action of G on $\mathbb{R}^n$. In particular, chambers are also called* fundamental chambers.

7.2 Classification of finite reflection groups

Theorem 7.7(3) has the following elementary corollary.

Corollary 7.11. *Let C be a chamber of a finite reflection group, and let r_α and r_β be two generating reflections, where α and β are outward normals to walls of C. Then $(\alpha, \beta) \leq 0$. In other words, all angles of C are acute (or non-obtuse, depending on the agreement whether $\pi/2$ is acute or not).*

Definition 7.12. A system of vectors is *indecomposable* if it cannot be split into two subsets orthogonal to each other.

Lemma 7.13. *Let $\{e_i\}$ be a finite indecomposable system of vectors in $\mathbb{R}^n$ such that $(e_i, e_j) \leq 0$ for all $i \neq j$. Then either all e_i are linearly independent, or there exists a unique (up to scaling) linear dependence, and all its coefficients are positive.*

Proof. Assume $\{e_i\}$ are linearly dependent, and there is a linear dependence with some coefficients positive and some non-positive. Define index sets I and J so that coefficients of $e_i > 0$ are positive if $i \in I$ and non-positive if $j \in J$. We then can write

$$\sum_{i \in I} c_i e_i = \sum_{j \in J} c_j e_j,$$

where $c_i > 0$ and $c_j \geq 0$. Denote $\alpha = \sum\limits_{i \in I} c_i e_i$ and $\beta = \sum\limits_{j \in J} c_j e_j$. Then

$$(\alpha, \beta) = \sum_{i \in I, j \in J} c_i c_j (e_i, e_j).$$

Since $\alpha = \beta$, the value above is non-negative. At the same time, all c_i and c_j are non-negative, and all (e_i, e_j) are non-positive, so the product is non-positive. Therefore, we conclude that $(\alpha, \beta) = 0$, and thus $\alpha = \beta = 0$.

Take any $j \in J$, then $(\alpha, e_j) = (0, e_j) = 0$. At the same time, $0 = (\alpha, e_j) = (\sum\limits_{i \in I} c_i e_i, e_j) = \sum\limits_{i \in I} c_i (e_i, e_j)$. Since all $c_i > 0$, this implies that $(e_i, e_j) = 0$ for all $i \in I$. As this holds for every $j \in J$, we get a contradiction with indecomposability of $\{e_i\}$.

Now, assume that there are two positive linear dependencies $\sum c_i e_i = 0 = \sum a_i e_i$. Since all a_i and c_i are positive, we can scale them such that $a_1 = c_1$. If the dependencies are still distinct, then subtracting one dependence from another we get a new dependence with the coefficient before e_1 vanishing, which contradicts the statement we already proved.

□

Corollary 7.14. *If $\{e_i\}$ is a finite indecomposable system of vectors in $\mathbb{R}^n$ with $(e_i, e_j) \leq 0$ for $i \neq j$, then $\#\{e_i\} \leq n + 1$.*

Proof. Indeed, if there are $n + 2$ vectors, then there exists a linear dependence on any $n + 1$ of them, so there is a dependence with some coefficients vanishing, which contradicts Lemma 7.13. □

The next statement follows from the construction of the chambers.

Lemma 7.15. *Let C_0 be a chamber, and let α_i be outward normals to the walls of C_0. Then $C_0 = \{v \in \mathbb{R}^n \mid (v, \alpha_i) < 0\}$.*