SMB problems sheet 4: additional questions

Here are a few more practice questions on the some of the material.

1 Total differential (Exact and inexact differentials)

1. Find the total differential of
   (a) $x^2y + zx^2$
   (b) $\sin(x)e^y$
   (c) $\sin(x) + \cos(t)x^2$
   (d) $\sin(x^2y^3z)$

   where x, y, z, t are all variables.
2. Write down a criterion for checking if a differential is exact or not and check that it is satisfied for the total differentials you computed above.
3. Are the following differentials exact or inexact? For those which are exact, find a function which has this as its total differntial.
   (a) $e^{x+t}((1 + x)dx + dt)$
   (b) $e^{x+t}((1 + x)dx + xdt)$
   (c) $y^2zdx + 2xyzdy + xy^2dz$
   (d) $yzdx + 2xzdy + xydz$

2 Jacobian and change of variable questions

1. Find the Jacobian of the change of variables $x = uv, \quad y = u/v$. By using this change of variables find $\iint_R \frac{1}{y}\, dx\, dy$ where R is the region in the positive quadrant $x > 0, y > 0$ bounded by the four curves $y = x,\ y = x/4,\ y = 1/x,\ y = 4/x$.
2. Find the Jacobian for the change of variables $x = e^{s+t}, \quad y = e^{s-t}$. By using this change of variables find $\iint_R \frac{1}{xy}\, dx\, dy$ where R is the region in the positive quadrant $x > 0, y > 0$ bounded by the four curves $y = x,\ y = x/e^2,\ y = 1/x,\ y = e^2/x$.
3. [Harder] Evaluate the integral $\iint_R (\sinh^2 x + \cos^2 y) \sinh 2x \sin 2y\, dx\, dy$ over the region R in the positive quadrant $x > 0, y > 0$, bounded by the four curves $x = 0,\ y = 0,\ \sinh x \cos y = 1,\ \cosh x \sin y = 1$ using the change of varibales $u = \sinh x \cos y, \quad v = \cosh x \sin y$.

3 Vector calculus

1. Compute ∇f for
   (a) $f = \sin x e^y z$
   (b) $f = x^3 \sin y + zxy$
   (c) $f = \cosh x^2 y + z$.

   In each case show that grad f is irrotational.
2. Compute $\nabla \cdot \mathbf{v}$ and $\nabla \times \mathbf{v}$ for
   (a) $\mathbf{v} = x\mathbf{i} + z\mathbf{j} + y\mathbf{k}$
   (b) $\mathbf{v} = x^2\mathbf{i} + \sin yx\mathbf{j} + zy\mathbf{k}$
   (c) $\mathbf{v} = \cosh x\mathbf{i} + zy\mathbf{j} + z\mathbf{k}$.

3. Show that $\mathbf{v} = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$ is irrotational, and find a scalar field $f(x, y, z)$ such that $\mathbf{v} = \nabla f$. Can you find another such f?

4. For $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $r = |\mathbf{r}|$ compute

   (a) ∇r

   (b) $\nabla \cdot \mathbf{r}$

   (c) $\nabla \times \mathbf{r}$

   (d) $\nabla f(r)$

   (e) $\nabla \cdot (\mathbf{r} f(r))$

   (f) $\nabla \times (\mathbf{r} f(r))$

5. Use the identities $\nabla \cdot (\mathbf{a} \times \mathbf{b}) = (\nabla \times \mathbf{a}) \cdot \mathbf{b} - \mathbf{a} \cdot (\nabla \times \mathbf{b})$ and $\nabla \times \nabla f = 0$ from lectures to prove that $\nabla \cdot (\nabla f \times \nabla g) = 0$.

6. Prove all the grad div curl identities given in lectures.