SMB problems sheet 3: vector calculus

30. In each of the following, find the rate of change of the given function at the given point in the direction of the given vector $\mathbf{a}$.
(a) $f(x,y) = x^2y, \quad (-1,1), \quad \mathbf{a} = \mathbf{i} + 2\mathbf{j}$.
(b) $f(x,y) = \frac{x}{1+y}, \quad (0,0), \quad \mathbf{a} = \mathbf{i} - \mathbf{j}$.
(c) $f(x,y) = x^2 + y^2, \quad (1,-2), \quad \mathbf{a}$ makes a positive angle of $60°$ with the positive x-axis.
(d) $f(x,y,z) = xy^3z^2, \quad (2,-1,4), \quad \mathbf{a} = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$.
(e) $f(x,y,z) = \sqrt{xy}\sin z, \quad (4,9,\pi/4), \quad \mathbf{a} = 2\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$.

31. In each of the following, sketch the level curve of the given function f through the given point, and find the direction of its tangent at that point. Also, find the direction of ∇f at the given point and confirm that it is perpendicular to the tangent to the level curve at that point.
(a) $f(x,y) = y^2 - x^2$, at $(2,1)$; (b) $f(x,y) = 3x - 2y$, at $(-2,1)$; (c) $f(x,y) = xy$, at $(3,2)$.

32. Find the gradient of $\phi(x,y,z) = z\sin y - xz$ at the point $(2,\pi/2,-1)$. Starting at this point, in what direction is ϕ decreasing most rapidly? Find the derivative of ϕ in the direction of the vector $2\mathbf{i} + 3\mathbf{j}$.

33. The surface of a certain lake is represented by a region of the xy plane such that the depth (in feet) under the point corresponding to (x,y) is $d(x,y) = 300 - 2x^2 - 3y^2$. A boat on which you are sitting sinks at the point $(9,6)$; in what direction should you start swimming so that the depth will decrease most rapidly? In what direction will the depth remain the same?

34. Let $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$, and let r be the magnitude of that vector. For $f(x,y) = \ln r$, show that $\nabla f = \dfrac{\mathbf{r}}{r^2}$.

35. Let $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, the position vector of a point, and $r = \sqrt{x^2+y^2+z^2}$.
    (a) Show that $\nabla(r^n) = \dfrac{-n\mathbf{r}}{r^{n+2}}$, and that $\nabla(\mathbf{p}.\mathbf{r}) = \mathbf{p}$ for any fixed vector $\mathbf{p}$.
    (b) If u and v are two scalar functions of x, y and z, show that $\nabla(uv) = u\nabla v + v\nabla u$.
    (c) In electrostatics, the electric field vector is $\mathbf{E} = -\nabla\phi$, where ϕ is the electrostatic potential. The potential due to an electric dipole at the origin of the coordinate system is $\phi = \dfrac{\mathbf{p}.\mathbf{r}}{r^3}$, where $\mathbf{p}$ is a fixed vector called the dipole moment. Use parts (a) and (b) to show that
$$\mathbf{E} = \frac{3(\mathbf{p}.\mathbf{r})\mathbf{r}}{r^5} - \frac{\mathbf{p}}{r^3}.$$

91. Santa's sleigh has trajectory $\mathbf{r}(t) = (3t - t^3)\,\mathbf{i} + 4\,\mathbf{j} + (\sin\pi t)\,\mathbf{k}$. Find its velocity and acceleration, and find its speed at time $t = 1$.

92. Homer's acceleration is $\mathbf{a} = \mathbf{j} + \exp(t/2)\mathbf{k}$. His initial position and velocity are $\mathbf{0}$ and $\mathbf{i} + 2\mathbf{k}$ respectively. Determine his position $\mathbf{r}(t)$.

93. Defining as usual $r \equiv |\mathbf{r}| = \sqrt{\mathbf{r}\cdot\mathbf{r}}$, show that $\dot{r} = \mathbf{r}\cdot\dot{\mathbf{r}}/r$. Hence if $\mathbf{r}(t) = t\,\mathbf{i} - t^2\,\mathbf{j} + t^3\,\mathbf{k}$ write down $r(t)$ and $\dot{\mathbf{r}}(t)$ and find an expression for $\dot{r}(t)$. Check by explicit differentiation of $r(t)$. Also find $v(t) = |\dot{\mathbf{r}}(t)|$ to confirm that $v \neq \dot{r}$. Is it true that $\dot{v} = |\ddot{\mathbf{r}}|$?

94. Now Santa's sleigh (mass m) has trajectory $\mathbf{r}(t) = t\,\mathbf{i} + t^2\,\mathbf{j} + t^3\,\mathbf{k}$. Find its angular momentum $\mathbf{L}$ about $\mathbf{r} = 0$ and check explicitly that it is perpendicular to both $\mathbf{r}$ and $\dot{\mathbf{r}}$.

95. The Tooth Fairy's position as a function of time t is $\mathbf{r}(t) = t\,\mathbf{i} + t^2\,\mathbf{j}$. Find $r(t) \equiv |\mathbf{r}(t)|$ and hence show that his velocity in polar coordinates is $\mathbf{v} = \{(1+2t^2)\,\boldsymbol{e}_r + t\,\boldsymbol{e}_\theta\}/\sqrt{1+t^2}$. Find his acceleration also in polar coordinates.

96. The Easter Bunny moves in a plane with constant speed v_0 on a spiral curve given by $r(\theta) = r_0\exp(\theta\cot\alpha)$. Show that her velocity is $\mathbf{v} = v_0(\cos\alpha\,\boldsymbol{e}_r + \sin\alpha\,\boldsymbol{e}_\theta)$ and find her acceleration $\mathbf{a}$ in polar coordinates. Show that $\mathbf{v}\cdot\mathbf{a} = 0$ and find $|\mathbf{a}|$.

97. The integral $\int \mathbf{a}(t)\,dt$, with $\mathbf{a} = t^2\mathbf{i} + (1+t)\mathbf{j} + (1-t)^2\mathbf{k}$, has the value $\mathbf{i}+\mathbf{j}+\mathbf{k}$ at $t=1$, find its value at $t=2$.

98. If $\mathbf{a} = t^2\mathbf{i} + (1+t)\mathbf{j} + (1-t)^2\mathbf{k}$ and $\mathbf{b} = (1-t)\mathbf{i} + t^2\mathbf{j} + (1+t)\mathbf{k}$, evaluate

    (a) $\int_0^1 \mathbf{a}\cdot\mathbf{b}\,dt$

    (b) $\int_0^1 \mathbf{a}\times\mathbf{b}\,dt$

99. Calculate the grad of the following scalar fields:

    (a) $f(x,y,z) = x^2y^3z^4$

    (b) $f(x,y,z) = \sin x\cos(yz)$

    (c) $f(x,y,z) = x^2yz + xy^2z$

100. Calculate the divergence and curl of the following vector fields:

    (a) $\mathbf{f}(x,y,z) = x^2y^2\mathbf{i} + 2xyz\mathbf{j} + z^2\mathbf{k}$

    (b) $\mathbf{f}(x,y,z) = x^2y^2\mathbf{i} + y^2z^2\mathbf{j} + x^2z^2\mathbf{k}$

    (c) $\mathbf{f}(x,y,z) = \sin x\mathbf{i} + \cos y\mathbf{j} + \tan z\mathbf{k}$

101. Jupiter and Saturn have masses M_J and M_S, $\mathbf{r}$ is the separation vector of Saturn from Jupiter and $r = |\mathbf{r}|$. The gravitational potential is $\phi(\mathbf{r}) = \frac{GM_JM_S}{r}$ and the force on Saturn is $\mathbf{F} = \nabla\phi$. Show that $\mathbf{F} = -\frac{M_JM_SG\mathbf{r}}{r^3}$.

102. Suppose that two vector fields are given by $\mathbf{g}(\mathbf{r}) = \mathbf{r}$ and $\mathbf{h}(\mathbf{r}) = \omega\times\mathbf{r}$, where ω is a constant vector. Compute:

    (a) $\nabla\cdot\mathbf{g}$; $\quad\nabla\times\mathbf{g}$.

    (b) $\nabla\cdot\mathbf{h}$; $\quad\nabla\times\mathbf{h}$

103. $\mathbf{b}$ is a vector field.

    (a) Prove using the formulae for $\nabla\times(\phi\,\mathbf{b})$ and for $\nabla\times(\nabla\times\mathbf{b})$ given in lectures that $\nabla\times(\mathbf{b}(\nabla\cdot\mathbf{b})) + \mathbf{b}\times[\nabla\times(\nabla\times\mathbf{b})] + \mathbf{b}\times\nabla^2\mathbf{b} := (\nabla\cdot\mathbf{b})(\nabla\times\mathbf{b})$.

    (b) Prove that $\mathbf{b}\times(\nabla\times\mathbf{b}) = \nabla(b^2/2) - (\mathbf{b}\cdot\nabla)\mathbf{b}$ (Hint: consider just one component (the z-component only to begin with, the other components are similar.)

104. Show that $\nabla\times(\nabla\times\mathbf{a}) = \nabla(\nabla\cdot\mathbf{a}) - \nabla^2\mathbf{a}$ for $\mathbf{a} = x^2z\mathbf{i} + y^2z\mathbf{j} + xyz\mathbf{k}$.

105. Coordinates (α,β,R) with $-1\le\alpha,\beta\le 1$, $0\le R<1$ are related to Cartesian coordinates (x,y,z) via $x = R\alpha, y = R\beta$ and $z = (1-\alpha^2-\beta^2)^{1/2}R$, $R = |\mathbf{r}|$.

    (a) Express $\mathbf{r}$ in terms of α,β,R and $\mathbf{i},\mathbf{j},\mathbf{k}$.

    (b) Obtain the vectors $\mathbf{e}_i(=\partial\mathbf{r}/\partial\alpha,\dots)$ and hence show that the scale factors h_i are given by

    $$h_1 = \frac{R(1-\beta^2)^{1/2}}{(1-\alpha^2-\beta^2)^{1/2}}, h_2 = \frac{R(1-\alpha^2)^{1/2}}{(1-\alpha^2-\beta^2)^{1/2}}, h3 = 1.$$

    (c) Verify formally that the system is not an orthogonal one.

    (d) Show that the volume element of the coordinate system is

    $$dV = \frac{R^2 d\alpha d\beta dR}{(1-\alpha^2-\beta^2)^{1/2}}.$$