SMB problems sheet 1: partial differentiation

1. Find the partial derivatives f_x, f_y, f_{xx}, f_{yy}, f_{xy} and f_{yx} when $f(x,y)$ is
(i) $xy^2 + x^2y^3 + x^3y^4$, (ii) $\sqrt{x^2+y^2}$, (iii) $\sin(xy)$, (iv) $\arctan(y/x)$, (v) $\exp(xy^2)$.

2. Find the partial derivatives f_x, f_y and f_z when $f(x,y,z)$ is
(i) $xy + xz + yz$, (ii) $\sqrt{x^2+2y^2+3z^2}$, (iii) $\cosh^2(xyz)$, (iv) $\operatorname{arctanh}\left(\frac{1}{x+y+z}\right)$, (v) $z\exp(y/x)$.

3. For $f(x,y) = \sqrt{x^2+y^2}$, show that $x\dfrac{\partial f}{\partial x} + y\dfrac{\partial f}{\partial y} = f$.

4. For $f(x,y) = \ln\sqrt{x^2+y^2}$, show that $x\dfrac{\partial f}{\partial x} + y\dfrac{\partial f}{\partial y} = 1$.

5. Show that for $t > 0$, $u(x,t) = \dfrac{1}{\sqrt{t}}\exp\left(-\dfrac{x^2}{4t}\right)$ satisfies the partial differential equation $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial t}$.

6. Let $f(x,y) = u(x,y)e^{ax+by}$ where $u(x,y)$ is a function for which $\dfrac{\partial^2 u}{\partial x \partial y} = 0$. Find values of the constants a and b such that $\dfrac{\partial^2 f}{\partial x \partial y} - \dfrac{\partial f}{\partial x} - \dfrac{\partial f}{\partial y} + f = 0$.

7. For which values of the constants a, b and c does $u(x,y) = ax + by + c$ satisfy the equation $\left(\dfrac{\partial u}{\partial x}\right)^2 + \left(\dfrac{\partial u}{\partial y}\right)^2 = 9$?

8. For which values of the constants α and β does $u(x,t) = \sin(\alpha x)\sin(\beta t)$ satisfy the wave equation $u_{tt} = c^2 u_{xx}$?

9. For $f(x,y) = x^2 - y^2$, use the chain rule to find $\frac{df}{dt}$ along the curve $x = 2\cos t$, $y = 3\sin t$.

10. Let $f(x,y) = e^{xy^2}$. Use the chain rule to find the value of $\frac{df}{dt}$ at the point on the curve $x = t\cos t$, $y = t\sin t$ where $t = \frac{\pi}{2}$.

11. Let $f(x,y,z) = x^2e^{2y}\cos 3z$. Use the chain rule to find the value of $\frac{df}{dt}$ at the point on the curve $x = \cos t$, $y = \ln(t+2)$, $z = t$ where $t = 0$.

12. Compute the partial derivatives f_x, f_t, f_{xx}, f_{xt} and f_{tt} for each of the following functions.
(a) $f(x,t) = 2x^2(x-t)$, (b) $f(x,t) = \cos(x-t) + \sin(x+t)$, (c) $f(x,t) = x\sin(xt) + t\log(x/t)$.

13. If f is a function of x and y, where $x = e^u\cosh v$ and $y = e^u\sinh v$, show that

$$\frac{\partial f}{\partial u} = x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}, \quad \frac{\partial f}{\partial v} = y\frac{\partial f}{\partial x} + x\frac{\partial f}{\partial y} \quad \text{and}$$

$$\frac{\partial^2 f}{\partial u \partial v} - \frac{\partial f}{\partial v} = xy\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right) + (x^2+y^2)\frac{\partial^2 f}{\partial x \partial y}.$$

14. If f is a function of x and y, where $x = \frac{1}{2}(u^2 - v^2)$ and $y = uv$, show that

$$u\frac{\partial f}{\partial v} - v\frac{\partial f}{\partial u} = 2\left(x\frac{\partial f}{\partial y} - y\frac{\partial f}{\partial x}\right) \quad \text{and}$$

$$\frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v^2} = (u^2+v^2)\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right).$$

15. If f is a function of x and y, where $x = e^u \cos v$ and $y = e^u \sin v$, show that

$$\frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v^2} = e^{2u}\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right).$$

16. If $x = a + u + v$ and $y = b + cu - cv$, where a, b and c are constants, and V is a function of x and y, find V_x, V_y, V_{xx}, V_{xy} and V_{yy} in terms of the partial derivatives of V with respect to u and v.

17. Given that $V(r) = \alpha/r$, where $r^2 = x^2 + y^2 + z^2$ and α is a constant, evaluate

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

for $r \neq 0$.

18. Find the general solution $u(x, y)$ of each of the following partial differential equations:
(i) $\dfrac{\partial u}{\partial x} = 4x + \sin y$, (ii) $\dfrac{\partial^2 u}{\partial x^2} = xy$, (iii) $\dfrac{\partial^2 u}{\partial x \partial y} = x^2 + y^2$, (iv) $\dfrac{\partial^2 u}{\partial y^2} = -x^2 \sin(xy)$.

19. Find the solution $u(x, t)$ of the equation $u_{tt} = 0$ satisfying the initial conditions $u(x, 0) = x^2$, $u_t(x, 0) = -x$.

20. Remembering what you learned about solving linear ordinary differential equations of first order, solve the following problems.

    (i) Find the function $u(x, t)$ which satisfies the partial differential equation $xu_x + 3u = x^2$ and the boundary condition $u(1, t) = (t + 1)/5$.

    (ii) Find the general solution $z(x, y)$ of the equation $x\dfrac{\partial^2 z}{\partial x^2} - \dfrac{\partial z}{\partial x} = 0$.

21. Use the change of variables $s = x - y$ and $t = x - 2y$ to find the general solution of the partial differential equation

$$2\frac{\partial^2 f}{\partial x^2} + 3\frac{\partial^2 f}{\partial x \partial y} + \frac{\partial^2 f}{\partial y^2} = 0.$$

22. If $u = f(2x - y) + g(x - 2y)$ show that

$$2\frac{\partial^2 u}{\partial x^2} + 5\frac{\partial^2 u}{\partial x \partial y} + 2\frac{\partial^2 u}{\partial y^2} = 0.$$

23. For the equation $u_t = 3u_x$, use the change of variables $s = x + 3t$, $v = x - 3t$, to find the solution $u(x, t)$ which satisfies the initial condition $u(x, 0) = \exp(-x^2)$.

24. Expand $f(x, y) = y^2/x^3$ about the point $(1, -1)$ up to and including quadratic terms.

25. Expand $f(x, y) = \sin(xy)$ about the point $\left(1, \frac{\pi}{3}\right)$ up to and including quadratic terms.

26. Expand $f(x, y) = e^{xy}$ about the point $(2, 3)$ up to and including quadratic terms.

27. Find and classify the critical points of the following functions
(i) $f(x, y) = x^3 + y^3 - 3x - 12y + 20$, (ii) $f(x, y) = x^4 + 4x^2y^2 - 2x^2 + 2y^2 - 1$,
(iii) $f(x, y) = (x^2 + y^2)^2 - 8(x^2 - y^2)$, (iv) $f(x, y) = e^{x^2 + y^2 - 2x + 4}$.

28. The material used to make the bottom of a rectangular box is twice as expensive per unit area as the material used to make the top or side walls. What are the dimensions of the cheapest box of volume 12 cm^3?

29. The internal surface area of a rectangular box without a lid is 27 cm^2. What dimensions give the maximum volume?

77. Find the total differential of the following functions:

(a) $f(x, y) = xe^{x+2y}$
(b) $f(x, t) = x \sin(at)$
(c) $f(x, y, z) = \sin(xyz)$
(d) $f(x, y, z) = \sin(x^2 + y^2 + z^2)$.

78. Determine which of the following are exact differentials:

(a) $(4x + 3)y\, dx + x(2x + 1)\, dy$;
(b) $y \sin x\, dx + x \sin y\, dy$;
(c) $y^3(\ln x + 1)\, dx + 3xy^2 \ln x\;\, dy$;
(d) $y^3(\ln x + 1)\, dy + 3xy^2 \ln x\, dx$;
(e) $[x/(x^2 + y^2)]\, dy - [y/(x^2 + y^2)]\, dx$.
(f) $(y^2 + 3x^2)\, dx + 2xy\, dy$
(g) $(x^2 + 3y^2)\, dx + 2xy\, dy$

79. Show that $df = y(2 - x^2)dx + x(x + 2)dy$ is not an exact differential.

Find the differential equation that a function $g(x)$ must satisfy if $d\phi = g(x)df$ is to be an exact differential. Verify that $g(x) = e^{-x}$ is a solution of this equation and deduce the form of $\phi(x, y)$.

80. The Dieterici equation of state for a gas takes the form $PV = RT \exp(-\alpha/(VRT))$, in which α and R are constants. Calculate expressions for $(\partial P/\partial V)_T$, $(\partial V/\partial T)_P$, $(\partial T/\partial P)_V$, and show that their product is -1, as stated in lectures.

81. The temperature of a point (x, y, z) on a sphere of radius 3 is $T(x, y, z) = 1 + 2xy + 2yz$. Using Lagrange multipliers find the temperature of the hottest point on the surface of the sphere.

82. Given that the internal energy U satisfies $dU = TdS - PdV$ (where P is the pressure, V the volume, S the entropy and T the temperature) derive a Maxwell relation connecting $(\partial V/\partial T)_P$ and $(\partial S/\partial P)_T$.

83. The entropy $S(H, T)$, the magnetisation $M(H, T)$ and the internal energy $U(H, T)$ of a magnetic salt placed in a magnetic field of strength H at temperature T are connected by the equation $TdS = dU - HdM$. Prove that $(\partial M/\partial T)_H = (\partial S/\partial H)_T$.

For a particular salt $M(H, T) = M_0[1 - \exp(-H/T)]$. Show that, at fixed T, if H is increased from zero to a strength such that $M = 1/2M_0$, then the entropy S decreases by $M_0(1 - \ln 2)/2$.

84. The functions $f(x, t)$ and $F(x)$ are defined by $f(x, t) = \cos(xt)$, $F(x) = \int_0^x f(x, t)dt$. Verify by explicit calculation that $dF/dx = f(x, x) + \int_0^x \partial f(x, t)/\partial x dt$.

85. Find the derivative with respect to x of the integrals

(a) $I(x) = \int_0^1 \sinh(xt^2)/t dt$.
(b) $I(x) = \int_x^{3x} \exp(xt)/t dt$.
(c) $I(x) = \int_x^{2x} \cos(xt)/t dt$.

86. Given that $\int_0^\infty e^{-\alpha t} dt = \alpha^{-1}$, prove, without explicit integration, that $\int_0^\infty t^n e^{-\alpha t} dt = \frac{n!}{\alpha^{n+1}}$.

87. By considering $\int_0^\pi \sin(xy)dx$, show that $\int_0^\pi [\sin(xy) + xy \cos(xy)]dx = \pi \sin \pi y$.

For more questions on partial differentiation see Riley ch. 5. All except questions 5.20 ... 5.24 are appropriate.