SMB problems sheet 2: multiple-integration

54. Evaluate the following double integrals
(i) $\displaystyle\int_0^2\int_0^1 (x^2+y^2)\,dxdy$, (ii) $\displaystyle\int_0^{\frac{\pi}{2}}\int_0^4 x\cos y\,dxdy$, (iii) $\displaystyle\int_1^2\int_1^2 \frac{xy}{\sqrt{x^2+y^2}}\,dxdy$,
(iv) $\displaystyle\int_0^1\int_{-1}^{y+1} (xy-x)\,dxdy$, (v) $\displaystyle\int_0^1\int_0^x x\cos\pi y\,dydx$, (vi) $\displaystyle\int_0^1\int_0^y xye^{x^2}\,dxdy$.

55. Evaluate the following double integrals
(i) $\displaystyle\iint_R x^3y\,dA$ where R is the interior of the triangle with vertices $(0,0)$, $(1,0)$ and $(1,1)$;
(ii) $\displaystyle\iint_R \sqrt{xy}\,dA$ where R is the finite region enclosed by the curves $y=x^2$ and $y=x^3$.

56. Use a double integral to find the area of the interior of the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$.

57. Find the area of the bounded region determined by the curves $xy=6$ and $x+y=5$.

58. Evaluate the following double integrals by changing the order of integration
(i) $\displaystyle\int_0^{\frac{\pi}{2}}\int_x^{\frac{\pi}{2}} \frac{\sin y}{y}\,dydx$, (ii) $\displaystyle\int_0^1\int_{\sqrt{x}}^1 \sin\left(\frac{y^3+1}{2}\right)\,dydx$, (iii) $\displaystyle\int_0^1\int_{x^2}^1 \frac{x^3}{\sqrt{x^4+y^2}}\,dydx$.

59. Use polar coordinates to evaluate the following double integrals
(i) $\displaystyle\iint_R e^{-x^2-y^2}\,dA$ where R is the region bounded by the circle $x^2+y^2=1$;
(ii) $\displaystyle\iint_R \frac{x}{\sqrt{x^2+y^2}}\,dA$ where R is the region $x^2+y^2\le 9$, $x\ge 0$ and $y\ge 0$.

60. Evaluate the integral

$$\iint_A \left(\frac{x^2}{x^2+y^2}\right)\,dxdy,$$

where A (for annulus) is the region between the two circles $x^2+y^2=1$ and $x^2+y^2=4$.

61. Evaluate the following triple integrals
(i) $\displaystyle\int_0^1\int_0^{1-x}\int_0^{2-x} xyz\,dzdydx$, (ii) $\displaystyle\int_0^{\frac{\pi}{2}}\int_0^1\int_0^2 zr^2\sin\theta\,dzdrd\theta$,
(iii) $\displaystyle\int_0^{\pi}\int_0^{\frac{\pi}{4}}\int_0^{\sec\phi} \sin 2\phi\,d\rho d\phi d\theta$.

62. Evaluate the triple integral $\displaystyle\iiint_Q x\,dV$ where Q is the region bounded by the planes $x+y+z=1$, $x=0$, $y=0$ and $z=0$.

63. Evaluate the triple integral $\displaystyle\iiint_S \exp\left[(x^2+y^2+z^2)^{\frac{3}{2}}\right]\,dV$ where S is unit sphere centred at the origin.

64. Evaluate $\iiint_Q (x^2+y^2+z^2)^{-3/2}\,dV$, where Q is the region bounded by the spheres $x^2+y^2+z^2=a^2$ and $x^2+y^2+z^2=b^2$, with $a>b>0$.

65. Evaluate the triple integral $\iiint_Q (x+y)^2\,dV$ where Q is the solid hemisphere $z\geq 0$, $x^2+y^2+z^2\leq 4$.

66. Find the volume of the finite region bounded by the paraboloid $z=4-x^2-y^2$ and the xy-plane.

67. Find the volume cut off the paraboloid $x^2+y^2=hz$ by the plane $z=h$.

68. Find the volume of the solid bounded by the paraboloids $z=\frac{1}{4}(x^2+y^2)$ and $z=5-x^2-y^2$.

69. A new auditorium is built with a foundation in the shape of a quadrant of a circle of radius 50 feet. Therefore, the foundation forms a region R bounded by the graph of $x^2+y^2=50^2$ with $x\geq 0$ and $y\geq 0$. The sloping floor is modelled by the equation $z=(x+y)/5$, and the ceiling is modelled by the equation $z=20+xy/100$. A heating engineer needs to know the volume of the hall. Can you calculate it?

70. The sombrero surface is defined by the equation

$$z=\frac{\sin r}{r},\quad \text{where}\quad r=\sqrt{x^2+y^2}.$$

Find the volume of the central peak, above the plane $z=0$. Find the volume of the first trough, below the plane $z=0$. Show that successive ridges above the plane $z=0$ and troughs below that plane have the same volume.

88. Compute the Jacobian on changing variables from cartesian co-ordinates (x,y) to polar co-ordinates (r,θ) to show that the area element $dA=rdrd\theta$.

89. Compute the Jacobian on changing variables from cartesian co-ordinates (x,y,z) to spherical polar co-ordinates (r,θ,ϕ) to show that the area element $dA=r^2\sin\theta drd\theta d\phi$.

90. Find the centre of mass of the objects (assuming uniform density) in

   (i) Question 66

   (ii) Question 67

   (iii) Question 68