# RATIONAL CURVES ON COMPLETE INTERSECTIONS AND THE CIRCLE METHOD 

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#### Abstract

We study the geometry of the space of rational curves on smooth complete intersections of low degree, which pass through a given set of points on the variety. The argument uses spreading out to a finite field, together with an adaptation to function fields of positive characteristic of work by Rydin Myerson on the circle method. Our work also allows us to handle weak approximation for such varieties.


## Contents

1. Introduction ..... 1
2. Background on finite fields and function fields ..... 4
3. Singular series and singular integral ..... 14
4. Application of the circle method ..... 23
5. Arithmetic deductions ..... 35
6. Geometric deductions ..... 39
7. Low degree rational curves ..... 48
References ..... 53

## 1. Introduction

Geometry of rational curves. Let $d \geqslant 2$ be an integer and let $k$ be a field whose characteristic exceeds $d$ if it is positive. Let $X \subset \mathbb{P}^{n-1}$ be a smooth complete intersection over $k$, cut out by $R$ hypersurfaces of the same degree $d$. We will always assume that $n \geqslant R d$, so that $X$ is Fano. In this paper we shall be interested in the geometry of the moduli space of degree $e$ rational curves on $X$ that pass through a given set of points on $X$.

To begin with, let $\mathscr{M}_{0,0}(X, e)$ be the moduli space of degree $e$ rational curves on $X$. As explained by Harris, Roth and Starr [15, Lemma 4.2], basic deformation theory shows that every irreducible component of $\mathscr{M}_{0,0}(X, e)$ has dimension at least

$$
\mu(e, R)=e(n-R d)+n-R-4
$$

which we refer to as the expected dimension. In the special case $R=1$ of smooth degree $d$ hypersurfaces $X \subset \mathbb{P}^{n-1}$, the space $\mathscr{M}_{0,0}(X, e)$ has been the focus of a great

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deal of study. For $d=2$, Kim and Pandharipande [17, Cor. 1] have addressed the irreducibility and dimension of $\mathscr{M}_{0,0}(X, e)$. For $d=3$, similar results hold thanks to work of Coskun and Starr [10] if $n \geqslant 5$. Using a version of Bend-and-Break, the best result for generic degree $d$ hypersurfaces $X$ is due to Riedl and Yang [23], who establish that $\mathscr{M}_{0,0}(X, e)$ is irreducible and of the expected dimension $\mu(e, 1)$, provided that $n \geqslant d+3$. Recent work of Browning and Sawin [7, Thm. 1.1] achieves the same conclusion for any smooth hypersurface $X \subset \mathbb{P}^{n-1}$ of degree $d$, provided that $n>(2 d-1) 2^{d-1}$. The latter result is proved using analytic number theory and builds on an approach employed by Browning and Vishe [8]. The idea is to study the moduli space of maps $\operatorname{Mor}_{e}\left(\mathbb{P}^{1}, X\right)$, whose expected dimension is $\mu(e, R)+3$, since $\mathscr{M}_{0,0}(X, e)$ is obtained from $\operatorname{Mor}_{e}\left(\mathbb{P}^{1}, X\right)$ on taking into account an action by $\mathrm{PGL}_{2}$. To calculate the dimension of $\operatorname{Mor}_{e}\left(\mathbb{P}^{1}, X\right)$, one begins by working over a finite field $k=\mathbb{F}_{q}$ of characteristic $>d$ and counts the number of points on it that are defined over a finite extension of $\mathbb{F}_{q}$. Finally, one combines this count with the Lang-Weil estimate to control the irreducibility and the dimension. In this paper we shall use the analytic number theory point of view to treat complete intersections with $R \geqslant 1$, beginning with the following result.

Theorem 1.1. Let $d \geqslant 2$ and let $k$ be a field whose characteristic exceeds $d$ if it is positive. Let $X \subset \mathbb{P}^{n-1}$ be a smooth complete intersection over $k$, cut out by $R$ hypersurfaces of degree $d$. Assume that $e \geqslant 1$ and

$$
n \geqslant \begin{cases}d(d-1) 2^{d+1} R+R & \text { if } d \geqslant 3  \tag{1.1}\\ 33 R & \text { if } d=2 .\end{cases}
$$

Then $\mathscr{M}_{0,0}(X, e)$ is an irreducible locally complete intersection of dimension $\mu(e, R)$.
For comparison, it follows from work of Beheshti and Kumar [1, Thm. 7.3] that a similar statement holds for generic complete intersections under the assumption $d R<(2 n+R-1) / 3$, which is equivalent to $n>\frac{1}{2}(3 d R-R+1)$.

Let $X$ be a smooth complete intersection over a field $k$, as in the statement of Theorem 1.1, Let $B \subset \mathbb{P}^{1}$ be the closed subscheme formed from $p_{1}, \ldots, p_{b} \in \mathbb{P}^{1}$. Consider the map

$$
\begin{equation*}
\text { red }: \operatorname{Mor}_{e}\left(\mathbb{P}^{1}, X\right) \rightarrow \operatorname{Mor}(B, X) \tag{1.2}
\end{equation*}
$$

that is obtained by reducing to the subscheme $B$, where $\operatorname{Mor}_{e}\left(\mathbb{P}^{1}, X\right)$ is the moduli space of maps and $\operatorname{Mor}(B, X)$ is isomorphic to $X^{b}$. For any $y_{1}, \ldots, y_{b} \in X$, we define

$$
\begin{equation*}
\mathrm{M}_{e, b}=\operatorname{Mor}_{e}\left(\mathbb{P}^{1}, X ; p_{1}, \ldots, p_{b} ; y_{1}, \ldots, y_{b}\right) \tag{1.3}
\end{equation*}
$$

to be the moduli space of degree $e$ morphisms $g: \mathbb{P}^{1} \rightarrow X$ such that $g\left(p_{j}\right)=y_{j}$ for $1 \leqslant j \leqslant b$. Thus $\mathrm{M}_{e, b}$ is the fibre of the map (1.2) over $\left(y_{1}, \ldots, y_{b}\right) \in X^{b}$. (Note that $\mathrm{M}_{e, 0}=\operatorname{Mor}_{e}\left(\mathbb{P}^{1}, X\right)$ when $b=0$.) As explained by Debarre [11, § 2.3], viewed as a fibre of the reduction map, the expected dimension of $\mathrm{M}_{e, b}$ is

$$
\begin{aligned}
\operatorname{dim} \operatorname{Mor}_{e}\left(\mathbb{P}^{1}, X\right)-b \operatorname{dim} X & =\mu(e, R)+3-b(n-1-R) \\
& =e(n-R d)+(n-1-R)(1-b)
\end{aligned}
$$

In his lecture at the Banff workshop "Geometry via Arithmetic" in July 2021, Will Sawin raised the question of tackling the irreducibility of the space $\mathrm{M}_{e, b}$ and calculating its dimension, pointing out that methods from algebraic geometry haven't yet been made to yield this information, even for generic complete intersections $X$. This is the object of the following result.

Theorem 1.2. Let $d \geqslant 2$ and let $k$ be a field whose characteristic exceeds $d$ if it is positive. Let $X \subset \mathbb{P}^{n-1}$ be a smooth complete intersection over $k$, cut out by $R$ hypersurfaces of degree $d$. Let $p_{1}, \ldots, p_{b} \in \mathbb{P}^{1}$ and let $y_{1}, \ldots, y_{b} \in X$, for $b \geqslant 1$. Assume that $e \geqslant(d+1+R) b$ and (1.1) holds. Then $\mathrm{M}_{e, b}$ is irreducible and has the expected dimension.

When $d=3$ and $n \geqslant 10$, work of Mânzățeanu [21, Cor. 1.4] addresses the geometry of the space $\mathrm{M}_{e, 2}$ in the special case $R=1$ of hypersurfaces. Since $X$ is smooth, the Hessian covariant is non-vanishing. Assuming that $y_{1}, y_{2} \in X$ don't both vanish on the Hessian, and that $e \geqslant 19-\frac{2}{n-9}$, it is shown that $\mathrm{M}_{e, 2}$ is irreducible and of the expected dimension. Theorem 1.2 arrives at the same conclusion for any $e \geqslant 10$, provided that $n \geqslant 97$.

Arithmetic over function fields. Our work on the geometry of $\mathrm{M}_{e, b}$ goes via a spreading out argument, as in [7, 8, which will lead us to study the size of $\# \mathrm{M}_{e, b}\left(\mathbb{F}_{q}\right)$ for a suitable finite field $\mathbb{F}_{q}$. This in turn will be accessed through a version of the Hardy-Littlewood circle method over the function field $\mathbb{F}_{q}(t)$. Let $X \subset \mathbb{P}^{n-1}$ be a smooth complete intersection cut out by $R$ hypersurfaces of equal degree $d \geqslant 2$, all of which are defined over $\mathbb{F}_{q}$. In suitable circumstances we shall be able to count $\mathbb{F}_{q}(t)$-rational points on $X$ of bounded height via the circle method. Over $\mathbb{Q}$, this is carried out in a classic paper of Birch [2], which is capable of proving that the Hasse principle and weak approximation hold, provided that $n \geqslant R+R(R+1) 2^{d-1}(d-1)$, a result that has already been extended to $\mathbb{F}_{q}(t)$ by Lee [20]. In fact, as described in [13, Thms. 3.4 and 3.6], it follows from the Lang-Nagata-Tsen theorems that $X\left(\mathbb{F}_{q}(t)\right) \neq \emptyset$ whenever $n>R d^{2}$.

The particular version of the circle method we shall use is a function field analogue of the work recently carried out over $\mathbb{Q}$ by Rydin Myerson [24, 25]. It has the key feature that it only depends linearly on the number $R$ of equations, rather than quadratically as in the case of Birch's result. To be precise, if $n \geqslant d 2^{d} R+R$ and the system of forms $f_{1}, \ldots, f_{R}$ is suitably generic, then one can conclude the Hasse principle and weak approximation over $\mathbb{Q}$. (Note that the genericity condition on the system of forms can be removed when $d=2$ [24] and when $d=3$ [25].) When working in positive characteristic, it turns out that we can remove the genericity condition for every $d$, provided that the finite field is large enough and we make a stronger assumption on the number of variables.

Theorem 1.3. Let $d, n, R \in \mathbb{N}$ such that $d \geqslant 2$ and

$$
n> \begin{cases}d(d-1) 2^{d} R+R & \text { if } d \geqslant 3 \\ 17 R & \text { if } d=2\end{cases}
$$

Assume that $\mathbb{F}_{q}$ is a finite field such that $\operatorname{char}\left(\mathbb{F}_{q}\right)>d$ and $q \geqslant(d-1)^{n}$. Let $f_{1}, \ldots, f_{R} \in \mathbb{F}_{q}\left[x_{1}, \ldots, x_{n}\right]$ be forms of degree $d$ cutting out a smooth complete intersection $X \subset \mathbb{P}^{n-1}$. Then $X$ satisfies weak approximation over $\mathbb{F}_{q}(t)$.

Our assumption on the number of variables is more stringent than the bound $n \geqslant d 2^{d} R+R$ obtained by Rydin Myerson for generic forms, although it retains the feature that it is linear in the number $R$ of equations. This can be traced to the nature of the saving in the main new technical ingredient, provided in Theorem 2.5, which allows us to handle all forms defining a smooth complete intersection in the function field setting.

Low degree rational curves. Let us return to the setting of Theorem 1.1, specialised to the case $k=\mathbb{F}_{q}$, a finite field whose characteristic exceeds $d$. Under the assumptions of the theorem our work ensures that $\mathscr{M}_{0,0}(X, e)$ is an irreducible locally complete intersection of dimension $\mu(e, R)$, for each $e \geqslant 1$. In the setting $d=3$ and $R=1$ of smooth cubic hypersurfaces, Kollár [18] ask about an explicit lower bound on $e$, sufficient to ensure that $X$ contains a degree $e$ rational curve defined over $\mathbb{F}_{q}$. Building on the proof of Theorem 1.3, our final result does exactly this in the broader setting of complete intersections, provided that $q$ is sufficiently large.

Theorem 1.4. There exists a constant $c_{d, n}>0$ depending only on $d$ and $n$ such that the following holds. Let $d \geqslant 2$ and let $\mathbb{F}_{q}$ be a finite field such that char $\left(\mathbb{F}_{q}\right)>d$ and $q \geqslant c_{d, n}$. Let $X \subset \mathbb{P}^{n-1}$ be a smooth complete intersection over $\mathbb{F}_{q}$, cut out by $R$ hypersurfaces of degree $d$. Assume that (1.1) holds and $e \geqslant 2(d-1) R+3 d$. Then $\mathscr{M}_{0,0}(X, e)\left(\mathbb{F}_{q}\right) \neq \emptyset$.

Let $X \subset \mathbb{P}^{n-1}$ be a smooth cubic hypersurface over $\mathbb{F}_{q}$. In [18, Example 7.6], Kollár proves that for any $q \geqslant c_{3, n}$ and any point $y \in X\left(\mathbb{F}_{q}\right)$, there exists a $\mathbb{F}_{q}$-rational curve of degree at most 216 on $X$ passing through $y$. Although it will not be pursued here, it would be possible to adapt the proof of Theorem 1.4 to yield explicit conditions on $e$ under which $\mathrm{M}_{e, b}\left(\mathbb{F}_{q}\right) \neq \emptyset$, for any given $b \geqslant 1$ and any smooth complete intersection $X \subset \mathbb{P}^{n-1}$ for which (1.1) holds.

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## 2. Background on finite fields and function fields

2.1. Points on varieties over finite fields. The Lang-Weil estimate can be used to give an upper bound for the number of $\mathbb{F}_{q}$-points on a variety over a finite field $\mathbb{F}_{q}$. We will need a general version of this that features convenient explicit constants. The following bound is proved in [9, Lemma 2.1], but we include a full proof here for the sake of completeness.

Lemma 2.1. Let $X \subset \mathbb{A}^{n}$ be an affine variety of dimension $r$ and degree $\delta$. Then $\# X\left(\mathbb{F}_{q}\right) \leqslant \delta q^{r}$.

Proof. We argue by induction on $r$. When $r=0$ then $X$ consists of at most $\delta$ points and the lemma is trivial. Assume now that we have $r \geqslant 1$. Let $Z$ be an irreducible component of $X$, with $\operatorname{dim} Z \geqslant 1$. We shall show that there is an index $1 \leqslant i \leqslant n$ such that the intersection of $Z$ with the hyperplane $H_{\alpha}=\left\{x_{i}=\alpha\right\}$ satisfies

$$
\operatorname{dim}\left(Z \cap H_{\alpha}\right)<\operatorname{dim} Z
$$

for any $\alpha \in \mathbb{F}_{q}$. Suppose otherwise, so that there exists $\alpha_{1}, \ldots, \alpha_{n} \in \mathbb{F}_{q}$, such that $Z \subset\left\{x_{j}=\alpha_{j}\right\}$ for each $1 \leqslant j \leqslant n$. However, this means that $Z=\left\{\left(\alpha_{1}, \ldots, \alpha_{n}\right)\right\}$, which contradicts the assumption that $\operatorname{dim} Z \geqslant 1$. Since $Z \cap H_{\alpha}$ has dimension at most $\operatorname{dim} Z-1$ and degree at most $\operatorname{deg} Z$, it follows by the induction hypothesis that

$$
\# Z\left(\mathbb{F}_{q}\right) \leqslant \sum_{\alpha \in \mathbb{F}_{q}} \#\left(Z \cap H_{\alpha}\right)\left(\mathbb{F}_{q}\right) \leqslant \sum_{\alpha \in \mathbb{F}_{q}}(\operatorname{deg} Z) q^{\operatorname{dim} Z-1} \leqslant(\operatorname{deg} Z) q^{\operatorname{dim} Z}
$$

Therefore, if we denote by $Z_{1}, \ldots, Z_{u}$ the irreducible components of $X$, then we obtain

$$
\# X\left(\mathbb{F}_{q}\right) \leqslant \sum_{1 \leqslant i \leqslant u} \# Z_{i}\left(\mathbb{F}_{q}\right) \leqslant \sum_{1 \leqslant i \leqslant u}\left(\operatorname{deg} Z_{i}\right) q^{\operatorname{dim} Z_{i}} \leqslant \sum_{1 \leqslant i \leqslant u}\left(\operatorname{deg} Z_{i}\right) q^{\operatorname{dim} X}
$$

Since $\operatorname{deg} X=\sum_{1 \leqslant i \leqslant u} \operatorname{deg} Z_{i}$, the result follows.
2.2. Function field notation. In this section we collect together some notation and basic facts concerning the function field $K=\mathbb{F}_{q}(t)$. Let $\Omega$ be the set of places of $K$. These correspond to either monic irreducible polynomials $\varpi$ in $\mathbb{F}_{q}[t]$, which we call the finite primes, or the prime at infinity $t^{-1}$ which we usually denote by $\infty$. The associated absolute value $|\cdot|_{v}$ is either $|\cdot|_{\infty}$ for some prime $\varpi \in \mathbb{F}_{q}[t]$ or $|\cdot|_{\infty}$, according to whether $v$ is a finite or infinite place, respectively. These are given by

$$
|a / b|_{\varpi}=\left(\frac{1}{q^{\operatorname{deg} \varpi}}\right)^{\operatorname{ord}_{\varpi}(a)-\operatorname{ord}_{\varpi}(b)} \quad \text { and } \quad|a / b|_{\infty}=q^{\operatorname{deg} a-\operatorname{deg} b}
$$

for any $a / b \in K^{*}$, where $\operatorname{ord}_{\varpi}(a)$ is the non-negative integer $\ell$ satisfying $\varpi^{\ell} \| a$. We extend these definitions to $K$ by taking $|0|_{\varpi}=|0|_{\infty}=0$. We will usually just write $|\cdot|=|\cdot|_{\infty}$.

For $v \in \Omega$ we let $K_{v}$ denote the completion of $K$ at $v$ with respect to $|\cdot|_{v}$, together with its ring of integers $\mathscr{O}_{v}$. We can extend the absolute value at the infinite place to $K_{\infty}$ to get a non-archimedean absolute value $|\cdot|: K_{\infty} \rightarrow \mathbb{R}_{\geqslant 0}$ given by $|\alpha|=q^{\text {ord } \alpha}$, where ord $\alpha$ is the largest $i \in \mathbb{Z}$ such that $a_{i} \neq 0$ in the representation $\alpha=\sum_{i \leqslant N} a_{i} t^{i}$. In this context we adopt the convention ord $0=-\infty$ and $|0|=0$. We extend this to vectors by setting $|\mathbf{x}|=\max _{1 \leqslant i \leqslant n}\left|x_{i}\right|$, for any $\mathbf{x} \in K_{\infty}^{n}$. It satisfies the ultrametric inequality $|\mathbf{x}+\mathbf{y}| \leqslant \max \{|\mathbf{x}|,|\mathbf{y}|\}$, for any $\mathbf{x}, \mathbf{y} \in K_{\infty}^{n}$. For given $\mathbf{x}, \mathbf{b} \in \mathbb{F}_{q}[t]^{n}$ and $m \in \mathbb{F}_{q}[t]$ we will write $\mathbf{x} \equiv \mathbf{b} \bmod m$ to mean that $\mathbf{x}=\mathbf{b}+m \mathbf{y}$ for some $\mathbf{y} \in \mathbb{F}_{q}[t]^{n}$.

We may identify $K_{\infty}$ with the set

$$
\mathbb{F}_{q}\left(\left(t^{-1}\right)\right)=\left\{\sum_{i \leqslant N} a_{i} t^{i}: \text { for } a_{i} \in \mathbb{F}_{q} \text { and some } N \in \mathbb{Z}\right\}
$$

and put

$$
\mathbb{T}=\left\{\alpha \in K_{\infty}:|\alpha|<1\right\}=\left\{\sum_{i \leqslant-1} a_{i} t^{i}: \text { for } a_{i} \in \mathbb{F}_{q}\right\}
$$

Since $\mathbb{T}$ is a locally compact additive subgroup of $K_{\infty}$ it possesses a unique Haar measure $\mathrm{d} \alpha$, which is normalised so that $\int_{\mathbb{T}} \mathrm{d} \alpha=1$. We can extend $\mathrm{d} \alpha$ to a (unique) translation-invariant measure on $K_{\infty}$ in such a way that

$$
\int_{\left\{\alpha \in K_{\infty}:|\alpha|<q^{N}\right\}} \mathrm{d} \alpha=q^{N},
$$

for any $N \in \mathbb{Z}$. These measures also extend to $\mathbb{T}^{n}$ and $K_{\infty}^{n}$, for any $n \in \mathbb{N}$.
Given $\alpha \in K_{\infty}$ we denote $\|\alpha\|=|\{\alpha\}|$, where $\{\alpha\} \in \mathbb{T}$ is the fractional part of $\alpha$. The following result will prove useful.

Lemma 2.2. For any $N \in \mathbb{Z}_{\geqslant 0}$ and any non-zero $h \in \mathbb{F}_{q}[t]$, we have

$$
\operatorname{meas}\left\{\beta \in \mathbb{T}:\|h \beta\|<q^{-N}\right\}=q^{-N} .
$$

Proof. We may suppose that $h=c_{s} t^{s}+\cdots+c_{0} \in \mathbb{F}_{q}[t]$, with $c_{s} \neq 0$ and $s \geqslant 0$. Note that

$$
\{h \beta\}=\sum_{i \leqslant-1}\left(c_{0} \beta_{i}+c_{1} \beta_{-1+i}+\cdots+c_{s} \beta_{-s+i}\right) t^{i}
$$

where we write $\beta=\sum_{i \leqslant-1} \beta_{i} t^{i}$. Then the restriction

$$
\|h \beta\|=|\{h \beta\}|<q^{-N}
$$

is equivalent to $\left(\beta_{-1}, \ldots, \beta_{-s-N}\right)$ satisfying

$$
\left(\begin{array}{cccccccc}
c_{0} & c_{1} & \cdots & \cdots & c_{s} & 0 & \cdots & 0 \\
0 & c_{0} & c_{1} & \cdots & \cdots & c_{s} & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & c_{0} & c_{1} & \cdots & \cdots & c_{s}
\end{array}\right)\left(\begin{array}{c}
\beta_{-1} \\
\beta_{-2} \\
\vdots \\
\beta_{-s-N}
\end{array}\right)=\mathbf{0} .
$$

Since $c_{s} \neq 0$, the matrix on the left hand side has rank $N$. Therefore, there are precisely $q^{s}$ choices for $\left(\beta_{-1}, \ldots, \beta_{-s-N}\right)$. It follows that

$$
\operatorname{meas}\left\{\beta \in \mathbb{T}:\|h \beta\|<q^{-N}\right\}=q^{s} \text { meas }\left\{\gamma \in \mathbb{T}:|\gamma|<q^{-s-N}\right\}=q^{-N}
$$

as required.
2.3. Characters. There is a non-trivial additive character $e_{q}: \mathbb{F}_{q} \rightarrow \mathbb{C}^{*}$ defined for each $a \in \mathbb{F}_{q}$ by taking $e_{q}(a)=\exp (2 \pi i \operatorname{Tr}(a) / p)$, where $\operatorname{Tr}: \mathbb{F}_{q} \rightarrow \mathbb{F}_{p}$ denotes the trace map. This character induces a non-trivial (unitary) additive character $\psi: K_{\infty} \rightarrow \mathbb{C}^{*}$ by defining $\psi(\alpha)=e_{q}\left(a_{-1}\right)$ for any $\alpha=\sum_{i \leqslant N} a_{i} t^{i}$ in $K_{\infty}$. We have the basic orthogonality property

$$
\sum_{\substack{b \in \mathbb{F}_{q}[t] \\ \text { and } \\|b|<q^{N}}} \psi(\gamma b)= \begin{cases}q^{N} & \text { if }|\gamma|<q^{-N} \\ 0 & \text { otherwise }\end{cases}
$$

for any $\gamma \in \mathbb{T}$ and any integer $N \geqslant 0$, as proved in [19, Lemma 7]. We also have

$$
\int_{\left\{\alpha \in K_{\infty}:|\alpha|<q^{N}\right\}} \psi(\alpha \gamma) \mathrm{d} \alpha= \begin{cases}q^{N} & \text { if }|\gamma|<q^{-N}  \tag{2.1}\\ 0 & \text { otherwise },\end{cases}
$$

for any $\gamma \in K_{\infty}$ and $N \in \mathbb{Z}$, as proved in [19, Lemma 1(f)].
2.4. Covering a box by boxes. Given $\mathbf{z} \in K_{\infty}^{R}$ and $N \in \mathbb{Z}$, we denote

$$
B_{N}(\mathbf{z})=\left\{\boldsymbol{\alpha} \in K_{\infty}^{R}:|\boldsymbol{\alpha}-\mathbf{z}|<q^{N}\right\} .
$$

Let $M \in \mathbb{Z}$ with $M \leqslant N$. We claim that we can always cover $B_{N}(\mathbf{z})$ by at most $q^{R(N-M)}$ boxes of the form $B_{M}(\mathbf{y})$. If $M=N$ there is nothing to prove. Suppose $M<N$ and assume without loss of generality that $\mathbf{z}=\mathbf{0}$. Then we have

$$
B_{N}(\mathbf{0})=\bigcup_{1 \leqslant i \leqslant R} \bigcup_{a_{M}^{(i)}, \ldots, a_{N-1}^{(i)} \in \mathbb{F}_{q}} B_{M}\left(\sum_{j=M}^{N-1} a_{j}^{(1)} t^{j}, \ldots, \sum_{j=M}^{N-1} a_{j}^{(R)} t^{j}\right)
$$

and the number of boxes used in this cover is precisely $q^{R(N-M)}$.
2.5. An auxiliary estimate. Let $f_{1}, \ldots, f_{R} \in \mathbb{F}_{q}\left[x_{1}, \ldots, x_{n}\right]$ be forms of degree $d$ defined over a finite field $\mathbb{F}_{q}$. We will assume that $\operatorname{char}\left(\mathbb{F}_{q}\right)>d$ throughout this section. Consider the affine variety

$$
\begin{equation*}
V=\left\{\mathbf{x} \in \mathbb{A}^{n}: \operatorname{rank}\left(\frac{\partial f_{k}}{\partial x_{i}}\right)_{\substack{1 \leqslant k \leqslant R \\ 1 \leqslant i \leqslant n}}<R\right\} . \tag{2.2}
\end{equation*}
$$

Note that $V$ cuts out the Birch singular locus occuring in the work of Birch [2]. As shown by Lee [20], the function field version of Birch's work allows one to count $\mathbb{F}_{q}(t)$-points of bounded height in the system $f_{1}=\cdots=f_{R}=0$, provided that $n-\operatorname{dim} V>R(R+1) 2^{d-1}(d-1)$.

Remark 2.3. Let $V$ be given by (2.2) and let $\sigma=\operatorname{dim} V$. If we assume that the system $f_{1}, \ldots, f_{R}$ cuts out a smooth complete intersection $X$ in $\mathbb{P}^{n-1}$, then we claim that $\sigma \leqslant R-1$. This follows from the argument in [5, Lemma 3.1] for optimal systems of forms, this being an argument that is valid over fields of arbitrary characteristic. In characteristic 0 , it is shown in [3, Lemma 3.1] that $X$ can be defined by an equivalent optimal system of forms, but use is made of Bertini's theorem, which is not generally available in positive characteristic. Fortunately, the same conclusion can be arrived at in positive characteristic, as explained by Glas in his proof of [12, Lemma 3.1].

Let

$$
\begin{equation*}
V_{\mathbf{h} . \mathbf{f}}=\left\{\mathbf{x} \in \mathbb{A}^{n}: h_{1} \nabla f_{1}(\mathbf{x})+\cdots+h_{R} \nabla f_{R}(\mathbf{x})=\mathbf{0}\right\} \tag{2.3}
\end{equation*}
$$

for any $\mathbf{h} \in \mathbb{F}_{q}[t]^{n}$. Then it will be convenient to define

$$
\begin{equation*}
\sigma_{\mathbf{f}}=\max _{\substack{\mathbf{h} \in \mathbb{E}_{q}[t] R^{R} \\ \mathbf{h} \neq 0}} \operatorname{dim} V_{\mathbf{h . f}} . \tag{2.4}
\end{equation*}
$$

Remark 2.4. Note that $V_{\text {h.f }}$ is contained in the variety $V$ defined in (2.2). Hence, if the system $f_{1}, \ldots, f_{R}$ cuts out a smooth complete intersection in $\mathbb{P}^{n-1}$, then Remark 2.3 implies that $\sigma_{\mathrm{f}} \leqslant \sigma \leqslant R-1$.

Suppose that

$$
f_{k}(\mathbf{x})=\sum_{i_{1}, \ldots, i_{d}=1}^{n} c_{i_{1}, \ldots, i_{d}}^{(k)} x_{i_{1}} \ldots x_{i_{d}}
$$

for $1 \leqslant k \leqslant R$, with coefficients $c_{i_{1}, \ldots, i_{d}}^{(k)} \in \mathbb{F}_{q}$ which are symmetric in the indices. Associated to each $f_{k}$ are the multilinear forms

$$
\begin{equation*}
\Psi_{i}^{(k)}\left(\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(d-1)}\right)=d!\sum_{i_{1}, \ldots, i_{d-1}=1}^{n} c_{i_{1}, \ldots, i_{d-1}, i}^{(k)} x_{i_{1}}^{(1)} \ldots x_{i_{d-1}}^{(d-1)} \tag{2.5}
\end{equation*}
$$

for $1 \leqslant i \leqslant n$. Given $J \in \mathbb{N}, \boldsymbol{\beta}=\left(\beta^{(1)}, \ldots, \beta^{(R)}\right) \in K_{\infty}^{R}$, we shall be interested in the size of the counting function

$$
N^{\operatorname{aux}}(J ; \boldsymbol{\beta})=\#\left\{\begin{array}{ll}
\underline{\mathbf{u}} \in \mathbb{F}_{q}[t]^{(d-1) n}: & \left|\mathbf{u}^{(1)}\right|, \ldots,\left|\mathbf{u}^{(d-1)}\right|<q^{J}  \tag{2.6}\\
& \left|\sum_{k=1}^{R} \beta^{(k)} \Psi_{i}^{(k)}(\underline{\mathbf{u}})\right|<q^{(d-2) J} \\
\text { for } 1 \leqslant i \leqslant n
\end{array}\right\}
$$

where $\underline{\mathbf{u}}=\left(\mathbf{u}^{(1)}, \ldots, \mathbf{u}^{(d-1)}\right)$. (This is a function field variant of the quantity considered in [24, Def. 1.1], in which the upper bound $|\boldsymbol{\beta}| q^{(d-2) J}$ is replaced by $q^{(d-2) J}$.) The trivial bound for this quantity is

$$
N^{\text {aux }}(J ; \boldsymbol{\beta}) \leqslant q^{J n(d-1)} .
$$

Our main result in this section offers the following improvement for suitable $\boldsymbol{\beta} \in K_{\infty}^{R}$.
Theorem 2.5. Let $J \in \mathbb{N}, M \in \mathbb{Z}$ and let $\sigma=\operatorname{dim} V$, where $V$ is the variety cut out by (2.2). Let $\boldsymbol{\beta} \in K_{\infty}^{R}$ such that $|\boldsymbol{\beta}|=q^{M}$. Assume that $M \geqslant d-2$. Then

$$
N^{\operatorname{aux}}(J ; \boldsymbol{\beta}) \leqslant(d-1)^{\min \left\{\frac{J+M}{d-1}, J\right\} n+n} q^{J n(d-1)-\min \left\{\frac{J+M}{d-1}, J\right\}(n-\sigma)} .
$$

Proof. Let $M^{*} \geqslant 0$ and define

$$
N^{\text {aux }}\left(J, M^{*} ; \boldsymbol{\beta}\right)=\#\left\{\underline{\mathbf{u}} \in \mathbb{F}_{q}[t]^{(d-1) n}: \quad \begin{array}{ll} 
& \left|\mathbf{u}^{(1)}\right|, \ldots,\left|\mathbf{u}^{(d-1)}\right|<q^{J} \\
& \left|\sum_{k=1}^{R} \beta^{(k)} \Psi_{i}^{(k)}(\underline{\mathbf{u}})\right|<q^{(d-2) J-M^{*}} \\
\text { for } 1 \leqslant i \leqslant n
\end{array}\right\},
$$

for any $\boldsymbol{\beta} \in K_{\infty}^{R}$ such that $|\boldsymbol{\beta}|=1$. Assuming that $M^{*}$ lies in the range

$$
\begin{equation*}
d-2 \leqslant M^{*} \leqslant J(d-2) \tag{2.7}
\end{equation*}
$$

we shall prove that

$$
\begin{equation*}
N^{\text {aux }}\left(J, M^{*} ; \boldsymbol{\beta}\right) \leqslant(d-1)^{\frac{\left(J+M^{*}\right) n}{d-1}+n} q^{J n(d-1)-\frac{\left(J+M^{*}\right)(n-\sigma)}{d-1}} . \tag{2.8}
\end{equation*}
$$

This will suffice for the theorem, since

$$
N^{\text {aux }}(J ; \boldsymbol{\beta}) \leqslant N^{\text {aux }}\left(J, \min \{M, J(d-2)\} ; \boldsymbol{\beta} t^{-M}\right)
$$

on recalling that $|\boldsymbol{\beta}|=q^{M}$.

Assume henceforth that $|\boldsymbol{\beta}|=1$ and let

$$
\beta^{(k)}=\sum_{r \geqslant 0} b_{r}^{(k)} t^{-r},
$$

for $1 \leqslant k \leqslant R$ and $b_{r}^{(k)} \in \mathbb{F}_{q}$. We may assume without loss of generality that $b_{0}^{(1)} \neq 0$, since $|\boldsymbol{\beta}|=1$. Since $\left|\mathbf{u}^{(j)}\right|<q^{J}$, for $1 \leqslant j \leqslant d-1$, each component of $\mathbf{u}^{(j)}$ can be written

$$
u_{i}^{(j)}=\sum_{\ell=0}^{J-1} z_{i, \ell}^{(j)} t^{\ell}
$$

for $1 \leqslant i \leqslant n$ and $z_{i, \ell}^{(j)} \in \mathbb{F}_{q}$. Expanding everything out, it therefore follows that $\sum_{k=1}^{R} \beta^{(k)} \Psi_{i}^{(k)}(\underline{\mathbf{u}})$ is equal to

$$
d!\sum_{k=1}^{R} \sum_{r \geqslant 0} b_{r}^{(k)} t^{-r} \sum_{i_{1}, \ldots, i_{d-1}=1}^{n} c_{i_{1}, \ldots, i_{d-1}, i}^{(k)} \sum_{\ell_{1}, \ldots, \ell_{d-1}=0}^{J-1} z_{i_{1}, \ell_{1}}^{(1)} \ldots z_{i_{d-1}, \ell_{d-1}}^{(d-1)} t^{\ell_{1}+\cdots+\ell_{d-1}}
$$

for $1 \leqslant i \leqslant n$. The components $z_{i, \ell}^{(j)}$ are therefore counted by $N^{\text {aux }}\left(J, M^{*} ; \boldsymbol{\beta}\right)$ if and only if the coefficient of $t^{(d-2)} J-M^{*}+v$ vanishes in this expression for all

$$
0 \leqslant v \leqslant J+M^{*}-(d-1)
$$

Note that this interval is non-empty, since $J \geqslant 1$ and $M^{*} \geqslant d-2$, by the lower bound in (2.7), whence $J+M^{*} \geqslant d-1$.

By collecting together the coefficients of $t^{(d-2) J-M^{*}+v}$, for $0 \leqslant v \leqslant J+M^{*}-(d-1)$, we see that $N^{\text {aux }}(J ; \boldsymbol{\beta})$ is equal to the number of elements $z_{i, \ell}^{(j)} \in \mathbb{F}_{q}$, for $1 \leqslant i \leqslant n$, $0 \leqslant \ell \leqslant J-1$ and $1 \leqslant j \leqslant d-1$, such that

$$
\sum_{k=1}^{R} \sum_{i_{1}, \ldots, i_{d-1}=1}^{n} c_{i_{1}, \ldots, i_{d-1}, i}^{(k)} \sum_{\substack{\ell_{1}, \ldots, \ell_{d-1}=0 \\ \ell_{1}+\cdots+\ell_{d-1} \geqslant v+(d-2) J-M^{*}}}^{J-1} b_{\ell_{1}+\cdots+\ell_{d-1}-(d-2) J-v+M^{*}}^{(k)} z_{i_{1}, \ell_{1}}^{(1)} \ldots z_{i_{d-1}, \ell_{d-1}}^{(d-1)}=0
$$

for $1 \leqslant i \leqslant n$ and $0 \leqslant v \leqslant J+M^{*}-(d-1)$. In total we have $J n(d-1)$ variables and $n\left(J+M^{*}-d+2\right)$ equations. We let $\widetilde{Z} \subset \mathbb{A}^{J(d-1) n}$ be the algebraic variety cut out by this system. For each choice of $v$, we let $X_{v} \subset \mathbb{A}^{J(d-1) n}$ denote the algebraic variety defined by the system of $n$ equations associated to it.

Let $g=\left\lfloor\left(J+M^{*}\right) /(d-1)\right\rfloor \geqslant 1$. We are unable to extract anything useful from many of the equations, but we will be able to exploit information when $v=$ $J+M^{*}-h(d-1)$ for integers $1 \leqslant h \leqslant g$, and also when $v=J+M^{*}-g(d-1)-1$, if $\left(J+M^{*}\right) /(d-1) \notin \mathbb{Z}$. For this purpose we define

$$
Z=X_{J+M^{*}-(d-1)} \cap X_{J+M^{*}-2(d-1)} \cap \cdots \cap X_{J+M^{*}-g(d-1)} \cap X_{J+M^{*}-g(d-1)-\epsilon_{J}},
$$

where

$$
\epsilon_{J}= \begin{cases}0 & \text { if } d-1 \mid J+M^{*} \\ 1 & \text { otherwise }\end{cases}
$$

In order to establish (2.8), it will suffice to prove that

$$
\begin{equation*}
\operatorname{dim} Z \leqslant J n(d-1)-\left(g+\epsilon_{J}\right)(n-\sigma) \tag{2.9}
\end{equation*}
$$

Note that our assumption (2.7) on $M^{*}$ implies that

$$
\left(g+\epsilon_{J}\right)(n-\sigma) \leqslant\left(\frac{J+M^{*}}{d-1}+1-\frac{1}{d-1}\right)(n-\sigma) \leqslant \operatorname{Jn}(d-1)
$$

so that the right hand side of (2.9) is indeed non-negative.
It follows from Bézout's theorem, in the form [14, Example 8.4.7], that the degree of $Z$ satisfies $\operatorname{deg} Z \leqslant(d-1)^{\left(g+\epsilon_{J}\right) n}$. Noting that $\widetilde{Z} \subset Z$, (2.8) follows from an application of Lemma 2.1 and the observation that

$$
\frac{J+M^{*}}{d-1} \leqslant g+\epsilon_{J} \leqslant \frac{J+M^{*}}{d-1}+1
$$

It therefore remains to prove (2.9).
The case $g=1$. In the definition of $X_{J+M^{*}-(d-1)}$, only the indices $\ell_{1}=\cdots=\ell_{d-1}=$ $J-1$ are possible. It follows that $X_{J+M^{*}-(d-1)}$ is given by the system of equations

$$
G_{i}\left(\mathbf{z}_{J-1}^{(1)}, \ldots, \mathbf{z}_{J-1}^{(d-1)}\right)=0
$$

for $1 \leqslant i \leqslant n$, where

$$
\begin{equation*}
G_{i}\left(\mathbf{z}_{J-1}^{(1)}, \ldots, \mathbf{z}_{J-1}^{(d-1)}\right)=\sum_{k=1}^{R} b_{0}^{(k)} \sum_{i_{1}, \ldots, i_{d-1}=1}^{n} c_{i_{1}, \ldots, i_{d-1}, i^{\prime}}^{(k)} z_{i_{1}, J-1}^{(1)} \ldots z_{i_{d-1}, J-1}^{(d-1)} . \tag{2.10}
\end{equation*}
$$

If $\varepsilon_{J}=1$ then, in the definition of $X_{J+M^{*}-(d-1)-1}$, we see that either $\ell_{1}=\cdots=$ $\ell_{d-1}=J-1$, or else precisely one of $\ell_{1}, \ldots, \ell_{d-1}$ is equal to $J-2$, with all the others equal to $J-1$. It follows that $X_{J+M^{*}-(d-1)-1}$ is given by the system of equations

$$
A_{i}\left(\mathbf{z}_{J-1}^{(1)}, \ldots, \mathbf{z}_{J-1}^{(d-1)}\right)+B_{i}\left(\mathbf{z}_{J-2}^{(1)}, \ldots, \mathbf{z}_{J-2}^{(d-1)}, \mathbf{z}_{J-1}^{(1)}, \ldots, \mathbf{z}_{J-1}^{(d-1)}\right)=0
$$

for $1 \leqslant i \leqslant n$, where

$$
\begin{equation*}
A_{i}\left(\mathbf{z}_{J-1}^{(1)}, \ldots, \mathbf{z}_{J-1}^{(d-1)}\right)=\sum_{k=1}^{R} b_{1}^{(k)} \sum_{i_{1}, \ldots, i_{d-1}=1}^{n} c_{i_{1}, \ldots, i_{d-1}, i^{\prime}}^{(k)} z_{i_{1}, J-1}^{(1)} \ldots z_{i_{d-1}, J-1}^{(d-1)} \tag{2.11}
\end{equation*}
$$

and $B_{i}\left(\mathbf{z}_{J-2}^{(1)}, \ldots, \mathbf{z}_{J-2}^{(d-1)}, \mathbf{z}_{J-1}^{(1)}, \ldots, \mathbf{z}_{J-1}^{(d-1)}\right)$ is given by

$$
\begin{equation*}
\sum_{m=1}^{d-1} \sum_{k=1}^{R} \sum_{i_{1}, \ldots, i_{d-1}=1}^{n} c_{i_{1}, \ldots, i_{d-1}, i}^{(k)} b_{0}^{(k)} z_{i_{1}, J-1}^{(1)} \ldots z_{i_{m-1}, J-1}^{(m-1)} z_{i_{m}, J-2}^{(m)} z_{i_{m+1}, J-1}^{(m+1)} \ldots z_{i_{d-1}, J-1}^{(d-1)} \tag{2.12}
\end{equation*}
$$

Let $Y_{J-1} \subset \mathbb{A}^{n(d-1)}$ be the variety that is obtained by projecting $X_{J+M^{*}-(d-1)}$ to the $\left(\mathbf{z}_{J-1}^{(1)}, \ldots, \mathbf{z}_{J-1}^{(d-1)}\right)$ variables, being defined by $G_{i}\left(\mathbf{z}_{J-1}^{(1)}, \ldots, \mathbf{z}_{J-1}^{(d-1)}\right)=0$, for $1 \leqslant$ $i \leqslant n$. We shall prove (2.9), for $g=1$, by projecting to $Y_{J-1}$, which we denote by $\pi_{0}: Z \rightarrow Y_{J-1}$, and considering the dimension of the fibres. The particular version of
the fibre dimension theorem we use is the Corollary to Theorem 2 in Mumford [22, §I.8]. Thus

$$
\begin{equation*}
\operatorname{dim} Z \leqslant \operatorname{dim} \overline{\pi_{0}(Z)}+\Delta_{0} \leqslant \operatorname{dim} Y_{J-1}+\Delta_{0} \tag{2.13}
\end{equation*}
$$

where $\Delta_{0}$ is the dimension of the fibre above any chosen point in the image of $\pi_{0}$. We claim that

$$
\begin{equation*}
\operatorname{dim} Y_{J-1} \leqslant n(d-2)+\sigma \tag{2.14}
\end{equation*}
$$

To see this, we follow Birch [2, Lemma 3.3] and proceed by intersecting the system of equations defining $Y_{J-1}$ with the diagonal

$$
D=\left\{\left(\mathbf{z}_{J-1}^{(1)}, \ldots, \mathbf{z}_{J-1}^{(d-1)}\right) \in \mathbb{A}^{n(d-1)}: \mathbf{z}_{J-1}^{(1)}=\cdots=\mathbf{z}_{J-1}^{(d-1)}\right\}
$$

This produces an algebraic variety cut out by the system

$$
\sum_{k=1}^{R} b_{0}^{(k)} \frac{\partial f_{k}(\mathbf{z})}{\partial z_{i}}=0
$$

for $1 \leqslant i \leqslant n$. Since $b_{0}^{(1)} \neq 0$ these relations imply that z must satisfy (2.2). Hence the affine dimension theorem implies that

$$
\sigma=\operatorname{dim} V \geqslant \operatorname{dim}\left(Y_{J-1} \cap D\right) \geqslant \operatorname{dim} Y_{J-1}+\operatorname{dim} D-n(d-1) .
$$

The diagonal has dimension $n$ and so it follows that (2.14) holds, as claimed.
It remains to bound $\Delta_{0}$. Suppose first that $\varepsilon_{J}=0$. Then we define the variety

$$
T_{J-1}=\left\{\left(\mathbf{z}_{0}^{(1)}, \ldots, \mathbf{z}_{J-1}^{(d-1)}\right) \in \mathbb{A}^{J n(d-1)}: \mathbf{z}_{J-1}^{(1)}=\cdots=\mathbf{z}_{J-1}^{(d-1)}=\mathbf{0}\right\} .
$$

It follows that

$$
\Delta_{0} \leqslant \operatorname{dim}\left(Z \cap T_{J-1}\right) \leqslant \operatorname{dim} T_{J-1} \leqslant(J-1) n(d-1)
$$

In this case, the desired bound (2.9) follows immediately from (2.13) and (2.14).
Suppose next that $\varepsilon_{J}=1$. Fix a point $\mathbf{w} \in \mathbb{A}^{n}$, to be determined in due course. Then we define the variety

$$
S_{J-1}(\mathbf{w})=\left\{\left(\mathbf{z}_{0}^{(1)}, \ldots, \mathbf{z}_{J-1}^{(d-1)}\right) \in \mathbb{A}^{J n(d-1)}: \mathbf{z}_{J-1}^{(1)}=\cdots=\mathbf{z}_{J-1}^{(d-2)}=\mathbf{w}, \mathbf{z}_{J-1}^{(d-1)}=\mathbf{0}\right\} .
$$

Note that $G_{i}(\mathbf{w}, \ldots, \mathbf{w}, \mathbf{0})=A_{i}(\mathbf{w}, \ldots, \mathbf{w}, \mathbf{0})=0$, for $1 \leqslant i \leqslant n$. It follows that

$$
\Delta_{0} \leqslant \operatorname{dim}\left(Z \cap S_{J-1}(\mathbf{w})\right)=\operatorname{dim}\left(X_{J+M^{*}-(d-1)-1} \cap S_{J-1}(\mathbf{w})\right)
$$

and from (2.11) and (2.12) that $X_{J+M^{*}-(d-1)-1} \cap S_{J-1}(\mathbf{w})$ is cut out by the system of equations

$$
\sum_{k=1}^{R} \sum_{i_{1}, \ldots, i_{d-1}=1}^{n} c_{i_{1}, \ldots, i_{d-1}, i}^{(k)} b_{0}^{(k)} w_{i_{1}} \ldots w_{i_{d-2}} z_{i_{d-1}, J-2}^{(d-1)}=0
$$

for $1 \leqslant i \leqslant n$. But this is just the system of equations

$$
H_{\mathbf{b} . \mathbf{f}}(\mathbf{w}) \mathbf{z}_{J-2}^{(d-1)}=\mathbf{0}
$$

where b.f $=b_{0}^{(1)} f_{1}+\cdots+b_{0}^{(R)} f_{R}$ and $H_{f}$ is the Hessian matrix of second derivatives associated to a polynomial $f \in \mathbb{F}_{q}\left[x_{1}, \ldots, x_{n}\right]$. Suppose that $H_{\text {b. } \mathbf{f}}(\mathbf{w})$ has rank $\varrho$. Then, on returning to $X_{J+M^{*}-(d-1)-1} \cap S_{J-1}(\mathbf{w})$, we see that there are no constraints
on the vectors $\mathbf{z}_{0}^{(1)}, \ldots, \mathbf{z}_{J-2}^{(d-2)}$, but $\mathbf{z}_{J-2}^{(d-1)}$ is constrained to lie in a linear space of dimension $n-\varrho$. Thus

$$
\Delta_{0} \leqslant(J-1) n(d-1)-\varrho .
$$

We now want to choose $\mathbf{w}$ to make $\varrho$ as large as possible. Let $T_{\varrho}$ be the variety of $\mathbf{w} \in \mathbb{A}^{n}$ such that $\operatorname{rank} H_{\mathbf{b} . \mathrm{f}}(\mathbf{w}) \leqslant \varrho$. Then it follows from precisely the same argument in [4, Lemma 2], together with Remark [2.4, that

$$
\operatorname{dim} T_{\varrho} \leqslant \varrho+\sigma
$$

since $b_{0}^{(1)} \neq 0$. Let $\varrho$ be the least non-negative integer such that $\operatorname{dim} T_{\varrho}=n$. We then choose $\mathbf{w} \in T_{\varrho} \backslash T_{\varrho-1}$, and therefore deduce that

$$
\Delta_{0} \leqslant(J-1) n(d-1)-(n-\sigma) .
$$

The desired bound (2.9) now follows from (2.13) and (2.14).
The case $g \geqslant 2$. It will be convenient to define the variety

$$
T_{i}=\left\{\left(\mathbf{z}_{0}^{(1)}, \ldots, \mathbf{z}_{J-1}^{(d-1)}\right) \in \mathbb{A}^{J n(d-1)}: \mathbf{z}_{i}^{(1)}=\cdots=\mathbf{z}_{i}^{(d-1)}=\mathbf{0}\right\},
$$

for each $i \in\{0, \ldots, J-1\}$. Let $Y_{J-2} \subset \mathbb{A}^{n(d-1)}$ be the variety obtained by projecting $X_{J+M^{*}-2(d-1)} \cap T_{J-1}$ to the $\left(\mathbf{z}_{J-2}^{(1)}, \ldots, \mathbf{z}_{J-2}^{(d-1)}\right)$ variables. Then we may define the projection

$$
\pi_{1}:\left(Z \cap T_{J-1}\right) \rightarrow Y_{J-2}
$$

Note that $X_{J+M^{*}-2(d-1)} \cap T_{J-1}$ is defined by the system of equations

$$
\sum_{k=1}^{R} \sum_{\substack{i_{1}, \ldots, i_{d-1}=1}}^{n} c_{i_{1}, \ldots, i_{d-1}, i}^{(k)} \sum_{\substack{\ell_{1}, \ldots, \ell_{d-1}=0 \\ \ell_{1}+\cdots+\ell_{d-1} \geqslant(J-2)(d-1)}}^{J-2} b_{\ell_{1}+\cdots+\ell_{d-1}-(J-2)(d-1)}^{(k)} z_{i_{1}, \ell_{1}}^{(1)} \ldots z_{i_{d-1}, \ell_{d-1}}^{(d-1)}=0,
$$

for $1 \leqslant i \leqslant n$. The summation conditions force $\ell_{1}=\cdots=\ell_{d-1}=J-2$ in this expression, so that we have the system $G_{i}\left(\mathbf{z}_{J-2}^{(1)}, \ldots, \mathbf{z}_{J-2}^{(d-1)}\right)=0$, for $1 \leqslant i \leqslant n$, in the notation of (2.10). Hence $Y_{J-2}$ is defined by exactly the same set of equations that was used to define the variety $Y_{J-1}$, from which it follows that $\operatorname{dim} Y_{J-2}=\operatorname{dim} Y_{J-1}$. By the same fibre dimension theorem as above, we obtain

$$
\operatorname{dim}\left(Z \cap T_{J-1}\right) \leqslant \operatorname{dim} Y_{J-1}+\Delta_{1}
$$

where $\Delta_{1}$ is the dimension of the fibre above any point in $\pi_{1}\left(Z \cap T_{J-1}\right)$. Taking the point

$$
\left(\mathbf{z}_{J-2}^{(1)}, \ldots, \mathbf{z}_{J-2}^{(d-1)}\right)=(\mathbf{0}, \ldots, \mathbf{0}) \in \pi_{1}\left(Z \cap T_{J-1}\right),
$$

we see that $\Delta_{1} \leqslant \operatorname{dim}\left(Z \cap T_{J-2} \cap T_{J-1}\right)$. Continuing inductively in this way, we obtain the bound

$$
\begin{aligned}
\operatorname{dim} Z & \leqslant \operatorname{dim} Y_{J-1}+\operatorname{dim}\left(T_{J-1} \cap Z\right) \\
& \leqslant 2 \operatorname{dim} Y_{J-1}+\operatorname{dim}\left(T_{J-2} \cap T_{J-1} \cap Z\right) \\
& \vdots \\
& \leqslant(g-1) \operatorname{dim} Y_{J-1}+\operatorname{dim}\left(T_{J-(g-1)} \cap \cdots \cap T_{J-1} \cap Z\right)
\end{aligned}
$$

Since $M^{*} \leqslant J(d-2)$, it follows that $J-g \geqslant 0$. If $\epsilon_{J}=0$ then we repeat this once more to deduce that

$$
\begin{aligned}
\operatorname{dim} Z & \leqslant g \operatorname{dim} Y_{J-1}+\operatorname{dim}\left(T_{J-g} \cap \cdots \cap T_{J-1} \cap \mathbb{A}^{n J(d-1)}\right) \\
& =g \operatorname{dim} Y_{J-1}+(J-g) n(d-1) .
\end{aligned}
$$

But then

$$
\operatorname{dim} Z \leqslant(J-g) n(d-1)+g(n(d-2)+\sigma)=J n(d-1)-g(n-\sigma),
$$

by (2.14), which is satisfactory for (2.9).
On the other hand, if $\epsilon_{J}=1$ then we deduce that

$$
\operatorname{dim}\left(T_{J-(g-1)} \cap \cdots \cap T_{J-1} \cap Z\right) \leqslant \operatorname{dim} Y_{J-1}+\Delta_{g}
$$

where $\Delta_{g}$ is the dimension of the fibre above any point in the image of the projection

$$
\pi_{g}: Z \cap T_{J-1} \cap \cdots \cap T_{J-(g-1)} \rightarrow Y_{J-g}
$$

where $Y_{J-g} \subset \mathbb{A}^{n(d-1)}$ is obtained by projecting $X_{J+M^{*}-g(d-1)} \cap T_{J-1} \cap \cdots \cap T_{J-(g-1)}$ to the $\left(\mathbf{z}_{J-g}^{(1)}, \ldots, \mathbf{z}_{J-g}^{(d-1)}\right)$ variables. We argue as in the case $g=1$, by picking a point $\mathbf{w} \in \mathbb{A}^{n}$ similarly to before and deducing that

$$
\begin{aligned}
\Delta_{g} & \leqslant \operatorname{dim}\left(Z \cap T_{J-1} \cap \cdots \cap T_{J-(g-1)} \cap S_{J-g}(\mathbf{w})\right) \\
& \leqslant \operatorname{dim}\left(X_{J+M^{*-g(d-1)-1}} \cap T_{J-1} \cap \cdots \cap T_{J-(g-1)} \cap S_{J-g}(\mathbf{w})\right),
\end{aligned}
$$

where

$$
S_{J-g}(\mathbf{w})=\left\{\left(\mathbf{z}_{0}^{(1)}, \ldots, \mathbf{z}_{J-1}^{(d-1)}\right) \in \mathbb{A}^{J n(d-1)}: \mathbf{z}_{J-g}^{(1)}=\cdots=\mathbf{z}_{J-g}^{(d-2)}=\mathbf{w}, \mathbf{z}_{J-g}^{(d-1)}=\mathbf{0}\right\} .
$$

Thus $\Delta_{g} \leqslant(J-g) n(d-1)-(n-\sigma)$, as previously, from which it follows that

$$
\operatorname{dim} Z \leqslant g \operatorname{dim} Y_{J-1}+(J-g) n(d-1)-(n-\sigma) \leqslant J n(d-1)-(g+1)(n-\sigma)
$$

by (2.14). This too is satisfactory for (2.9), which thereby completes the proof of Theorem 2.5

Corollary 2.6. Assume that the system $f_{1}, \ldots, f_{R}$ cuts out a smooth complete intersection in $\mathbb{P}^{n-1}$. Let $J \in \mathbb{N}$ and let $\boldsymbol{\beta} \in K_{\infty}^{R}$ such that $|\boldsymbol{\beta}| \geqslant q^{d-1}$. Then

$$
N^{\operatorname{aux}}(J ; \boldsymbol{\beta}) \leqslant(d-1)^{n} q^{J n(d-1)}\left(q^{n\left(1-\log _{q}(d-1)\right)-R+1}\right)^{-\theta_{d}(J)}
$$

where

$$
\theta_{d}(J)= \begin{cases}\frac{J}{d-1}+1 & \text { if } d \geqslant 3 \text { and } J \geqslant 2 \\ 1 & \text { if } d \geqslant 3 \text { and } J=1 \\ J & \text { if } d=2\end{cases}
$$

Proof. Recall from Remark 2.3 that $\sigma \leqslant R-1$. Suppose that $|\boldsymbol{\beta}|=q^{M}$, where $M \geqslant d-1$. Theorem 2.5 then implies that

$$
N^{\operatorname{aux}}(J ; \boldsymbol{\beta}) \leqslant(d-1)^{n} q^{J n(d-1)}\left(\frac{(d-1)^{n}}{q^{n-R+1}}\right)^{\min \left\{\frac{J+M}{d-1}, J\right\}}
$$

The statement follows on noting that $d-1=q^{\log _{q}(d-1)}$ and

$$
\min \left\{\frac{J+M}{d-1}, J\right\} \geqslant \min \left\{\frac{J}{d-1}+1, J\right\}=\theta_{d}(J)
$$

under the assumption $M \geqslant d-1$.
2.6. A further technical estimate. Now that Theorem[2.5 is established, our final result will also prove useful. Let $\boldsymbol{\beta}=\left(\beta^{(1)}, \ldots, \beta^{(R)}\right) \in K_{\infty}^{R}$. For each $0 \leqslant v \leqslant d-1$, let

$$
N^{(v)}(J ; \boldsymbol{\beta})=\#\left\{\begin{array}{ll} 
& \left|\mathbf{u}^{(1)}\right|, \ldots,\left|\mathbf{u}^{(v)}\right|<q^{J}  \tag{2.15}\\
\underline{\mathbf{u}} \in \mathbb{F}_{q}[t]^{(d-1) n}: & \left|\mathbf{u}^{(v+1)}\right|, \ldots,\left|\mathbf{u}^{(d-1)}\right|<q^{P} \\
& \left\|\sum_{k=1}^{R} \beta^{(k)} \Psi_{i}^{(k)}(\underline{\mathbf{u}})\right\|<q^{-(v+1) P+v J} \\
\text { for all } 1 \leqslant i \leqslant n
\end{array}\right\},
$$

where $\underline{\mathbf{u}}=\left(\mathbf{u}^{(1)}, \ldots, \mathbf{u}^{(d-1)}\right)$.
Lemma 2.7. Let $J, P \in \mathbb{N}$ such that $J \leqslant P$ and let $\boldsymbol{\beta} \in K_{\infty}^{R}$. Then precisely one of the following two alternatives must happen:
(i) we have

$$
q^{-J} \leqslant \max \left\{q^{-d P+d-2}|\boldsymbol{\beta}|^{-1},|\boldsymbol{\beta}|^{\frac{1}{d-1}} q^{-1}\right\} ;
$$

(ii) we have

$$
q^{J-d P+d-1} \leqslant|\boldsymbol{\beta}| \leqslant q^{-J(d-1)+d-2}
$$

and $N^{(d-1)}(J ; \boldsymbol{\beta})=N^{\text {aux }}\left(J ; \boldsymbol{\beta} t^{d P-J}\right)$, in the notation of (2.6).
Proof. Suppose that alternative (i) fails, so that $|\boldsymbol{\beta}|$ satisfies the inequalities in alternative (ii). Then

$$
\left|\sum_{k=1}^{R} \beta^{(k)} \Psi_{i}^{(k)}(\underline{\mathbf{u}})\right| \leqslant|\boldsymbol{\beta}| q^{(d-1)(J-1)} \leqslant q^{d-2} \cdot q^{-(d-1)}<1,
$$

for any $\left|\mathbf{u}^{(1)}\right|, \ldots,\left|\mathbf{u}^{(d-1)}\right|<q^{J}$. But then the statement clearly follows.

## 3. Singular series and singular integral

Let $f_{1}, \ldots, f_{R} \in \mathbb{F}_{q}\left[x_{1}, \ldots, x_{n}\right]$ be forms of degree $d$ defined over a finite field $\mathbb{F}_{q}$. We will assume that the system of equations $f_{1}=\cdots=f_{R}=0$ defines a smooth complete intersection in $\mathbb{P}^{n-1}$. Let $m \in \mathbb{F}_{q}[t]$ be a monic polynomial and let $\mathbf{b} \in \mathbb{F}_{q}[t]^{n}$ be such that $\operatorname{gcd}(\mathbf{b}, m)=1$ and $f_{i}(\mathbf{b}) \equiv 0 \bmod m$, for $1 \leqslant i \leqslant R$. We define

$$
\begin{equation*}
\mathbf{F}(\mathbf{x})=\mathbf{f}(m \mathbf{x}+\mathbf{b}), \tag{3.1}
\end{equation*}
$$

where we write $\mathbf{f}=\left(f_{1}, \ldots, f_{R}\right)$ for the vector of polynomials, and similarly for $\mathbf{F}$. In Section 4 we shall initiate the application of the circle method. The purpose of this section is to discuss, in isolation, the singular series and singular integral that will arise in our analysis. In traditional applications of the circle method it is customary to recycle estimates from the minor arcs in order to establish the convergence of the singular series and the singular integral. In the setting of smooth complete
intersections defined over the constant field $\mathbb{F}_{q}$, however, a direct treatment is more efficient.
3.1. Singular series. Given a non-zero polynomial $g \in \mathbb{F}_{q}[t]$ and $\mathbf{a} \in \mathbb{F}_{q}[t]^{R}$, we define

$$
\begin{equation*}
S_{g}(\mathbf{a})=|g|^{-n} \sum_{|\mathbf{y}|<|g|} \psi\left(\frac{\mathbf{a} \cdot \mathbf{F}(\mathbf{y})}{g}\right) \tag{3.2}
\end{equation*}
$$

where $\mathbf{F}$ is given in (3.1) and $\psi$ is defined in Section 2.3. In this section we study the singular series

$$
\mathfrak{S}=\sum_{\substack{g \in \mathbb{F}_{q}[t] \backslash\{0\}  \tag{3.3}\\
g \text { monic }}} \sum_{\substack{\mathbf{a} \in \mathbb{F}_{q}[t]^{R} \\
\left\lvert\, \begin{array}{l}
\text { a } \\
\text { gcd } \mid \text { gm } \\
\operatorname{gcd}(\mathbf{a}, g)=1
\end{array}\right.}} S_{g m^{d}}(\mathbf{a}),
$$

with the aim of establishing its convergence under relatively mild hypotheses. The singular series is slightly awkward to work with, since the inner sum sum over a runs modulo $\mathrm{gm}^{d}$, but only over those a which are coprime to $g$.

Define

$$
\begin{equation*}
A(g)=\sum_{\substack{\mathbf{a} \in \mathbb{F}_{q}[]|R\\| \mathbf{a}| ||g| \\ \operatorname{gcd}(\mathbf{a}, g)=1}} S_{g}(\mathbf{a}), \tag{3.4}
\end{equation*}
$$

for any monic $g \in \mathbb{F}_{q}[t]$. This is exactly the sum $\mathscr{A}(g)$ that appears in the work of Lee [20, Eq. (4.6.24)], and the multiplicativity of $A(g)$ follows from [20, Lemma 4.7.2]. Moreover, it is easy to check that

$$
\begin{equation*}
\sum_{\substack{\mathbf{a} \in \mathbb{F}_{q}[t]^{R} \\|\mathbf{a}|<\left|g m^{d}\right| \\ \operatorname{gcd}(\mathbf{a}, g)=1}} S_{g m^{d}}(\mathbf{a})=\sum_{\substack{h \mid m^{d} \\ \operatorname{gcd}(h, g)=1}} A\left(g m^{d} / h\right) . \tag{3.5}
\end{equation*}
$$

For now, we proceed to collect together some facts about $A(g)$ for an arbitrary monic $g \in \mathbb{F}_{q}[t]$.

Proposition 3.1. Let $d \geqslant 2$ and assume that $n \geqslant d R$. Suppose that $\operatorname{gcd}(\mathbf{b}, m)=1$ and $f_{i}(\mathbf{b}) \equiv 0 \bmod m$, for $1 \leqslant i \leqslant R$. Then there exists a constant $C>0$ that depends only on $d$ and $n$, such that

$$
|A(g)| \leqslant C^{\omega(g)}|g|^{-\left(1-\frac{1}{d}\right)\left(\frac{n}{d}-R\right)}|\operatorname{gcd}(g, m)|^{\left(1-\frac{1}{d}\right) \frac{n}{d}+\frac{R}{d}}
$$

By multiplicativity it suffices to bound $A(g)$ when $g=\pi^{e}$ is a prime power, with $e \geqslant 1$. To begin with, orthogonality of characters yields

$$
\begin{align*}
A\left(\pi^{e}\right) & =|\pi|^{-e n} \sum_{\substack{\mathbf{a} \in \mathbb{F}_{q}[t]^{R} \\
\left|\mathbf{a}<\left|e^{e}\right| \\
\operatorname{gcd}(\mathbf{a}, \pi)=1\right.}} \sum_{|\mathbf{y}|<\left|\pi^{e}\right|} \psi\left(\frac{\mathbf{a} \cdot \mathbf{F}(\mathbf{y})}{\pi^{e}}\right)  \tag{3.6}\\
& =|\pi|^{-e(n-R)}\left(N\left(\pi^{e}\right)-|\pi|^{n-R} N\left(\pi^{e-1}\right)\right),
\end{align*}
$$

where

$$
\begin{equation*}
N\left(\pi^{e}\right)=\#\left\{\mathbf{y} \in \mathbb{F}_{q}[t]^{n}:|\mathbf{y}|<\left|\pi^{e}\right|, F_{i}(\mathbf{y}) \equiv 0 \bmod \pi^{e} \text { for } 1 \leqslant i \leqslant R\right\} \tag{3.7}
\end{equation*}
$$

Recall that $F_{i}$ is defined in terms of $f_{i}$ via (3.1), for $1 \leqslant i \leqslant R$. We shall need to differentiate according to whether or not $\pi$ divides $m$.

Lemma 3.2. Let $d \geqslant 2$ and assume that $n \geqslant d R$. Assume that $e \geqslant 1$ and $\pi \nmid m$. Then

$$
A\left(\pi^{e}\right) \ll|\pi|^{-\left(1-\frac{1}{d}\right)\left(\frac{n}{d}-R\right) e}
$$

where the implied constant depends only on $d$ and $n$.
Proof. Since $\pi \nmid m$ we can make a non-singular change of variables $\mathbf{z}=m \mathbf{y}+\mathbf{b}$ modulo $\pi^{e}$, finding that

$$
N\left(\pi^{e}\right)=\#\left\{\mathbf{z} \in \mathbb{F}_{q}[t]^{n}:|\mathbf{z}|<\left|\pi^{e}\right|, f_{i}(\mathbf{z}) \equiv 0 \bmod \pi^{e} \text { for } 1 \leqslant i \leqslant R\right\}
$$

It will be convenient to also define

$$
N^{*}\left(\pi^{e}\right)=\#\left\{\mathbf{z} \in \mathbb{F}_{q}[t]^{n}:|\mathbf{z}|<\left|\pi^{e}\right|, \pi \nmid \mathbf{z}, f_{i}(\mathbf{z}) \equiv 0 \bmod \pi^{e} \text { for } 1 \leqslant i \leqslant R\right\}
$$

In view of the fact that $f_{1}=\cdots=f_{R}=0$ defines a smooth complete intersection in $\mathbb{P}^{n-1}$, Hensel lifting yields

$$
\begin{equation*}
N^{*}\left(\pi^{e}\right)=|\pi|^{n-R} N^{*}\left(\pi^{e-1}\right) \quad \text { if } e \geqslant 2 . \tag{3.8}
\end{equation*}
$$

Our analysis of $N\left(\pi^{e}\right)$ will depend on the relative size of $e$ to $d$. Suppose first that $e=1$. As explained in the appendix by Katz to [16, Deligne's resolution of the Riemann hypothesis now yields

$$
\begin{equation*}
N(\pi)=|\pi|^{n-R}+O\left(|\pi|^{\frac{n-R+1}{2}}\right) \tag{3.9}
\end{equation*}
$$

whence

$$
A(\pi)=|\pi|^{-(n-R)}\left(N(\pi)-|\pi|^{n-R}\right) \ll|\pi|^{-\frac{n-R-1}{2}},
$$

by (3.6), which is satisfactory for the lemma.
Suppose now that $2 \leqslant e \leqslant d$. Then we have

$$
N\left(\pi^{e}\right)=N^{*}\left(\pi^{e}\right)+|\pi|^{n(e-1)} .
$$

On appealing to (3.6) and (3.8), it therefore follows that

$$
A\left(\pi^{e}\right)=|\pi|^{e R-n}\left(1-|\pi|^{-R}\right)=O\left(|\pi|^{e R-n}\right)
$$

It is easy to check that $e R-n \leqslant-\left(1-\frac{1}{d}\right)\left(\frac{n}{d}-R\right) e$ if $e \leqslant d$ and $n \geqslant d R$. Thus this too is satisfactory for the lemma.

Finally we must deal with the case $e>d$. On sorting according to the $\pi$-adic valuation of $\mathbf{z}$, we obtain

$$
\begin{aligned}
N\left(\pi^{e}\right) & =\sum_{0 \leqslant j \leqslant e} \#\left\{\mathbf{z} \in \mathbb{F}_{q}[t]^{n}:|\mathbf{z}|<\left|\pi^{e}\right|, \operatorname{gcd}\left(\mathbf{z}, \pi^{e}\right)=\pi^{j}, \mathbf{f}(\mathbf{z}) \equiv \mathbf{0} \bmod \pi^{e}\right\} \\
& =\sum_{0 \leqslant j \leqslant e} \#\left\{\mathbf{z}^{\prime} \in \mathbb{F}_{q}[t]^{n}:\left|\mathbf{z}^{\prime}\right|<\left|\pi^{e-j}\right|, \operatorname{gcd}\left(\mathbf{z}^{\prime}, \pi^{e-j}\right)=1, \pi^{d j} \mathbf{f}\left(\mathbf{z}^{\prime}\right) \equiv \mathbf{0} \bmod \pi^{e}\right\} .
\end{aligned}
$$

When $d j \geqslant e$ we bound the inner cardinality by $|\pi|^{(e-j) n}$. When $d j<e$, the congruence only depends on the value of $\mathbf{z}^{\prime}$ modulo $\pi^{e-d j}$, whence

$$
\begin{equation*}
N\left(\pi^{e}\right)=\sum_{0 \leqslant j<e / d}|\pi|^{(d-1) j n} N^{*}\left(\pi^{e-d j}\right)+O\left(|\pi|^{\left(1-\frac{1}{d}\right) e n}\right), \tag{3.10}
\end{equation*}
$$

since $\sum_{e / d \leqslant j \leqslant e}|\pi|^{(e-j) n} \ll|\pi|^{\left(1-\frac{1}{d}\right) e n}$. Note that $n-R+\left(1-\frac{1}{d}\right)(e-1) n \geqslant\left(1-\frac{1}{d}\right) e n$. Returning to (3.6), we see that

$$
\begin{aligned}
|\pi|^{e(n-R)} A\left(\pi^{e}\right)= & N\left(\pi^{e}\right)-|\pi|^{n-R} N\left(\pi^{e-1}\right) \\
= & \sum_{0 \leqslant j<e / d}|\pi|^{(d-1) j n} N^{*}\left(\pi^{e-d j}\right) \\
& -|\pi|^{n-R} \sum_{0 \leqslant j<(e-1) / d}|\pi|^{(d-1) j n} N^{*}\left(\pi^{e-1-d j}\right)+O\left(|\pi|^{\frac{n}{d}-R+\left(1-\frac{1}{d}\right) e n}\right) .
\end{aligned}
$$

The first sum runs over $j$ such that $e-d j \geqslant 1$ and the second sum only involves $j$ with $e-d j \geqslant 2$. It follows from (3.8) that all terms cancel apart from a possible term with $e-d j=1$, provided that $d \mid e-1$. It now follows from (3.9) that

$$
\begin{aligned}
A\left(\pi^{e}\right) & \ll|\pi|^{-e(n-R)}\left(|\pi|^{(d-1) \frac{(e-1) n}{d}+n-R}+|\pi|^{\frac{n}{d}-R+\left(1-\frac{1}{d}\right) e n}\right) \\
& \ll|\pi|^{-e(n-R)} \cdot|\pi|^{\frac{n}{d}-R+\left(1-\frac{1}{d}\right) e n} \\
& =|\pi|^{-\left(\frac{n}{d}-R\right)(e-1)} .
\end{aligned}
$$

But $e-1>e\left(1-\frac{1}{d}\right)$ if $e>d$, whence $A\left(\pi^{e}\right) \ll|\pi|^{-\left(1-\frac{1}{d}\right)\left(\frac{n}{d}-R\right) e}$. This finally completes the proof of the lemma.

Lemma 3.3. Suppose that $\mu \in \mathbb{Z}$ is such that $\pi^{\mu} \| m$, with $\mu \geqslant 1$. Assume that $e \geqslant 1$ and $f_{i}(\mathbf{b}) \equiv 0 \bmod \pi^{\mu}$ for $1 \leqslant i \leqslant R$, with $\operatorname{gcd}(\mathbf{b}, \pi)=1$. Then

$$
A\left(\pi^{e}\right)=0 \quad \text { if } e>\mu
$$

and

$$
\left|A\left(\pi^{e}\right)\right| \leqslant 2|\pi|^{e R} \quad \text { if } e \leqslant \mu
$$

In particular, we have $\left|A\left(\pi^{e}\right)\right| \leqslant 2|\pi|^{\min \{e, \mu\} R}$.
Proof. It follows from (3.1) and (3.6) that

$$
A\left(\pi^{e}\right)=|\pi|^{-e(n-R)}\left(N\left(\pi^{e}\right)-|\pi|^{n-R} N\left(\pi^{e-1}\right)\right)
$$

where

$$
N\left(\pi^{e}\right)=\#\left\{\mathbf{y} \in \mathbb{F}_{q}[t]^{n}:|\mathbf{y}|<\left|\pi^{e}\right|, \mathbf{f}(m \mathbf{y}+\mathbf{b}) \equiv \mathbf{0} \bmod \pi^{e}\right\} .
$$

If $e \leqslant \mu$ we simply take

$$
\left|N\left(\pi^{e}\right)-|\pi|^{n-R} N\left(\pi^{e-1}\right)\right| \leqslant N\left(\pi^{e}\right)+|\pi|^{n-R} N\left(\pi^{e-1}\right) \leqslant 2|\pi|^{e n},
$$

whence $\left|A\left(\pi^{e}\right)\right| \leqslant 2|\pi|^{e R}$.

Suppose now that $e>\mu$ and write $m=\pi^{\mu} m^{\prime}$, where $m^{\prime} \in \mathbb{F}_{q}[t]$ is coprime to $\pi$. Then, for each $1 \leqslant i \leqslant R$, the congruence $f_{i}\left(\pi^{\mu} m^{\prime} \mathbf{y}+\mathbf{b}\right) \equiv 0 \bmod \pi^{e}$ only depends on the value of $\mathbf{y}$ modulo $\pi^{e-\mu}$. Hence

$$
\begin{aligned}
N\left(\pi^{e}\right) & =|\pi|^{\mu n} \#\left\{\mathbf{y} \in \mathbb{F}_{q}[t]^{n}:|\mathbf{y}|<\left|\pi^{e-\mu}\right|, \mathbf{f}\left(\pi^{\mu} m^{\prime} \mathbf{y}+\mathbf{b}\right) \equiv \mathbf{0} \bmod \pi^{e}\right\} \\
& =|\pi|^{\mu n} N^{\dagger}\left(\pi^{e}\right),
\end{aligned}
$$

where

$$
\begin{equation*}
N^{\dagger}\left(\pi^{e}\right)=\#\left\{\mathbf{z} \in \mathbb{F}_{q}[t]^{n}:|\mathbf{z}|<\left|\pi^{e}\right|, \mathbf{f}(\mathbf{z}) \equiv \mathbf{0} \bmod \pi^{e}, \mathbf{z} \equiv \mathbf{b} \bmod \pi^{\mu}\right\} \tag{3.11}
\end{equation*}
$$

We note that $e \geqslant 2$, since we are assuming that $\mu \geqslant 1$. Making the change of variables $\mathbf{z}=\mathbf{v}+\pi^{e-1} \mathbf{w}$, for $\mathbf{v}$ modulo $\pi^{e-1}$ and $\mathbf{w}$ modulo $\pi$, we deduce that

$$
N^{\dagger}\left(\pi^{e}\right)=\sum_{\substack{\mathbf{v} \in \mathbb{F}_{q}\left[t n^{n} \\
|\mathbf{v}|<\left|\pi^{e-1}\right| \\
\mathbf{f}\left(\mathbf{v}=\mathbf{0} \bmod \pi^{e-1} \\
\mathbf{v} \equiv \mathbf{b} \bmod \pi^{e}\right.\right.}} \#\left\{\begin{array}{l}
\left.\mathbf{w} \in \mathbb{F}_{q}[t]^{n}: \begin{array}{l}
|\mathbf{w}|<|\pi| \\
\mathbf{w} \cdot \nabla f_{i}(\mathbf{v}) \equiv-f_{i}(\mathbf{v}) / \pi^{e-1} \bmod \pi \\
\text { for } 1 \leqslant i \leqslant R
\end{array}\right\} . ~ . ~ . ~ . ~
\end{array}\right.
$$

Since $\operatorname{gcd}(\mathbf{b}, \pi)=1$, we have $\operatorname{gcd}(\mathbf{v}, \pi)=1$ in the outer sum. Thus the $R \times n$ matrix $\left(\nabla f_{1}(\mathbf{v}), \ldots, \nabla f_{R}(\mathbf{v})\right)$ has full rank, since $f_{1}=\cdots=f_{R}=0$ defines a smooth complete intersection in $\mathbb{P}^{n-1}$. We conclude that the inner quantity is precisely $|\pi|^{n-R}$, whence

$$
\begin{equation*}
N^{\dagger}\left(\pi^{e}\right)=|\pi|^{n-R} N^{\dagger}\left(\pi^{e-1}\right) \tag{3.12}
\end{equation*}
$$

We have therefore shown that

$$
A\left(\pi^{e}\right)=|\pi|^{-e(n-R)+\mu n}\left(N^{\dagger}\left(\pi^{e}\right)-|\pi|^{n-R} N^{\dagger}\left(\pi^{e-1}\right)\right)=0
$$

The statement of the lemma follows.
Proof of Proposition 3.1. We write $g=h g^{\prime}$ where $h=\operatorname{gcd}(g, m)$ and $g^{\prime}$ is coprime to $m$. Once combined with Lemmas 3.2 and 3.3 , it follows from the multiplicativity of $A(g)$ that

$$
\begin{aligned}
|A(g)| & =\prod_{\pi^{e} \| g^{\prime}}\left|A\left(\pi^{e}\right)\right| \prod_{\pi^{e} \| h}\left|A\left(\pi^{e}\right)\right| \\
& \leqslant C^{\omega(g)}\left|g^{\prime}\right|^{-\left(1-\frac{1}{d}\right)\left(\frac{n}{d}-R\right)}|h|^{R} \\
& =C^{\omega(g)}|g|^{-\left(1-\frac{1}{d}\right)\left(\frac{n}{d}-R\right)}|h|^{\left(1-\frac{1}{d}\right)\left(\frac{n}{d}-R\right)+R}
\end{aligned}
$$

for some constant $C>0$ that depends only on $d$ and $n$. Now clearly

$$
\left(1-\frac{1}{d}\right)\left(\frac{n}{d}-R\right)+R=\left(1-\frac{1}{d}\right) \frac{n}{d}+\frac{R}{d},
$$

and so the statement of Proposition 3.1 follows.
We may now return to the treatment of the singular series (3.3). We begin by proving the following result.

Lemma 3.4. We have

$$
\sum_{\substack{\mathbf{a} \in \mathbb{F}_{q}[]^{R} \\\left|\mathbf{a}<\left|\left|g m^{d}\right| \\ \operatorname{gcd}(\mathbf{a}, g)=1\right.\right.}} S_{g m^{d}}(\mathbf{a})=\mathbf{1}_{\operatorname{gcd}(g, m)=1}|m|^{R} A(g) .
$$

Proof. It follows from (3.5) that

$$
\sum_{\substack{\mathbf{a} \in \mathbb{F}_{q}[t]^{R} \\|\overrightarrow{ }|<\left|g m^{d}\right| \\ \operatorname{gcd}(\mathbf{a}, g)=1}} S_{g m^{d}}(\mathbf{a})=\sum_{\substack{h \mid m^{d} \\ \operatorname{gcd}(h, g)=1}} A(g \tilde{m}),
$$

where $\tilde{m}=m^{d} / h$. Next, we factorise $\tilde{m}=m_{0} m_{1}$, where

$$
m_{0}=\prod_{\substack{\pi^{\nu} \| \tilde{m} \\ \pi \mid g}} \pi^{\nu}, \quad m_{1}=\prod_{\substack{\pi^{\nu} \| \tilde{m} \\ \pi \nmid g}} \pi^{\nu}
$$

Then we may write $g \tilde{m}=g_{0} m_{1}$, where $g_{0}=g m_{0}$. In particular, $\operatorname{gcd}\left(m_{0}, m_{1}\right)=1$ and $\operatorname{gcd}\left(g_{0}, m_{1}\right)=1$, whence $A(g \tilde{m})=A\left(g_{0}\right) A\left(m_{1}\right)$, by multiplicativity. The first part of Lemma 3.3 implies that the first product vanishes as soon as $g$ shares a common prime factor with $m$. Therefore $m_{0}=1$ and we deduce that
since the condition $\operatorname{gcd}(h, g)=1$ is implied by the condition $\operatorname{gcd}(g, m)=1$ when $h \mid m^{d}$. But

$$
\sum_{h \mid m^{d}} A(\tilde{m})=\sum_{\substack{\mathbf{a} \in \mathbb{F}_{[ }[t]^{R} \\|\mathbf{a}|<\left|m^{d}\right|}} S_{m^{d}}(\mathbf{a})=\frac{N\left(m^{d}\right)}{|m|^{d(n-R)}},
$$

by orthogonality of characters. Using the Chinese remainder theorem we see that

$$
N\left(m^{d}\right)=\prod_{\pi^{\mu} \| m} N\left(\pi^{d \mu}\right)=\prod_{\pi^{\mu} \| m}|\pi|^{\mu n} N^{\dagger}\left(\pi^{d \mu}\right)
$$

in the notation of (3.11). Appealing to (3.12), it now follows that

$$
N\left(m^{d}\right)=|m|^{n} \prod_{\pi^{\mu} \| m}|\pi|^{(n-R)(d-1) \mu} N^{\dagger}\left(\pi^{\mu}\right)=|m|^{R+d(n-R)},
$$

since $N^{\dagger}(m)=1$. We deduce that

$$
\sum_{h \mid m^{d}} A(\tilde{m})=|m|^{R}
$$

from which the lemma follows.

We define the truncated singular series

$$
\begin{equation*}
\mathfrak{S}(B)=\sum_{\substack{g \in \mathbb{F}_{q}[t] \\ g \text { monic } \\ 0<|g| \leqslant q^{B}}} \sum_{\substack{\mathbf{a} \in \mathbb{F}_{q}[t]^{R} \\ \operatorname{scd}\left|<\left|g m^{d}\right| \\ \operatorname{scd}(\mathbf{a}, g)=1\right.}} S_{g m^{d}}(\mathbf{a}), \tag{3.13}
\end{equation*}
$$

for any $B \in \mathbb{Z}_{\geqslant 0}$. It follows from Lemma 3.4 that

$$
\begin{equation*}
\mathfrak{S}(B)=|m|^{R} \sum_{\substack{g \text { monic } \\ 0<|g| q^{B} \\ \operatorname{gcd}(g, m)=1}} A(g) . \tag{3.14}
\end{equation*}
$$

Formally extending the sum to infinity, the singular series (3.3) can also be written

$$
\begin{equation*}
\mathfrak{S}=|m|^{R} \sum_{\substack{g \text { monic } \\ \operatorname{gcd}(g, m)=1}} A(g) \tag{3.15}
\end{equation*}
$$

We shall need to establish the convergence of $\mathfrak{S}$, and show how well we can approximate it by $\mathfrak{S}(B)$. For this we are led to study the quantity

$$
\begin{equation*}
\mathscr{S}(T)=\sum_{\substack{g \text { monic } \\|g|=q^{T} \\ \operatorname{gcd}(g, m)=1}}|A(g)|, \tag{3.16}
\end{equation*}
$$

for any $T \in \mathbb{Z}_{\geqslant 0}$.
Lemma 3.5. Let $d \geqslant 2$ and assume that $n>d R$. Let $\varepsilon>0$. Suppose that $\operatorname{gcd}(\mathbf{b}, m)=1$ and $f_{i}(\mathbf{b}) \equiv 0 \bmod m$, for $1 \leqslant i \leqslant R$. Then, for any $T \in \mathbb{Z}_{\geqslant 0}$, we have

$$
\mathscr{S}(T) \ll_{\varepsilon} q^{T(1+\varepsilon)-T\left(1-\frac{1}{d}\right)\left(\frac{n}{d}-R\right)}
$$

Proof. We shall use Proposition 3.1 to prove this result. First, for any $\varepsilon>0$, it follows from the divisor sum bound in function fields that $C^{\omega(g)}=O_{\varepsilon}\left(|g|^{\varepsilon}\right)$, where the implied constant depends only on $\varepsilon$ and $C$. (Note that, as explained in [21, Lemma 2.3], the implied constant in this estimate is independent of $q$.) We deduce from Proposition 3.1 that

$$
\mathscr{S}(T) \lll \ll \begin{gathered}
g \in \mathbb{F}_{q}[t] \\
g \text { monic } \\
|g|=q^{T}
\end{gathered}|g|^{-\left(1-\frac{1}{d}\right)\left(\frac{n}{d}-R\right)+\varepsilon} \ll \varepsilon \varepsilon q^{T(1+\varepsilon)-T\left(1-\frac{1}{d}\right)\left(\frac{n}{d}-R\right)} .
$$

The statement of the lemma follows.
The following result summarises our treatment of the singular series $\mathfrak{S}$ and its approximation by the truncated singular series $\mathfrak{S}(B)$.

Lemma 3.6. Let $d \geqslant 2$ and assume that $\left(1-\frac{1}{d}\right)\left(\frac{n}{d}-R\right)>1$. Let $\varepsilon>0$. Suppose that $\operatorname{gcd}(\mathbf{b}, m)=1$ and $f_{i}(\mathbf{b}) \equiv 0 \bmod m$, for $1 \leqslant i \leqslant R$. Then, for any $B \in \mathbb{Z}_{\geqslant 0}$, we have

$$
|\mathfrak{S}-\mathfrak{S}(B)|<_{\varepsilon} q^{R \operatorname{deg} m+(B+1)\left(1+\varepsilon-\left(1-\frac{1}{d}\right)\left(\frac{n}{d}-R\right)\right)}
$$

Moreover, $\mathfrak{S}$ is absolutely convergent and satisfies $0<\mathfrak{S} \ll q^{R \operatorname{deg} m}$. The implied constants in these estimates depend only on $d, n, R$ and $\varepsilon$.

Proof. It follows from (3.14) and (3.15) that

$$
|\mathfrak{S}-\mathfrak{S}(B)| \leqslant q^{R \operatorname{deg} m} \sum_{k=B+1}^{\infty} \mathscr{S}(k)
$$

in the notation of (3.16). Since $\left(1-\frac{1}{d}\right)\left(\frac{n}{d}-R\right)>1$, Lemma 3.5 yields

$$
\begin{aligned}
|\mathfrak{S}-\mathfrak{S}(B)| & \lll q^{R \operatorname{deg} m} \sum_{k=B+1}^{\infty} q^{k(1+\varepsilon)-k\left(1-\frac{1}{d}\right)\left(\frac{n}{d}-R\right)} \\
& <_{\varepsilon} q^{R \operatorname{deg} m+(B+1)\left(1+\varepsilon-\left(1-\frac{1}{d}\right)\left(\frac{n}{d}-R\right)\right)}
\end{aligned}
$$

The first part of the lemma readily follows. Next, we may reapply Lemma 3.5 to deduce the upper bound $\mathfrak{S} \ll q^{R \operatorname{deg} m}$.

It remains to establish the positivity of $\mathfrak{S}$ under the assumptions of the lemma, having already established that $\mathfrak{S}$ is absolutely convergent. The argument in [20, Corollary 4.7.7] ensures that $\mathfrak{S}>0$ provided that there exists a smooth $\mathscr{O}_{\pi}$-point on the variety $f_{1}=\cdots=f_{R}=0$, for every finite prime $\pi \nmid m$. But this follows from the assumptions of the lemma, since the completion of $\mathbb{F}_{q}(t)$ at $\pi$ is a $C_{1}$ field.
3.2. Singular integral. For any $\gamma \in K_{\infty}^{R}$, we define

$$
\begin{equation*}
S_{\infty}(\boldsymbol{\gamma})=\int_{\mathbb{T}^{n}} \psi(\boldsymbol{\gamma} \cdot \mathbf{f}(\mathbf{v})) \mathrm{d} \mathbf{v} . \tag{3.17}
\end{equation*}
$$

In analogy to (3.16), let

$$
\begin{equation*}
\mathscr{I}(T)=\int_{|\gamma|=q^{T}} S_{\infty}(\boldsymbol{\gamma}) \mathrm{d} \boldsymbol{\gamma} \tag{3.18}
\end{equation*}
$$

for any $T \in \mathbb{Z}_{\geqslant 0}$. We begin by proving the following upper bound.
Lemma 3.7. Let $d \geqslant 2$ and assume that $n \geqslant d R$. For any $T \in \mathbb{Z}_{\geqslant 0}$ we have

$$
\mathscr{I}(T) \ll \min \left\{q^{R(T+1)}, q^{\left(1-\frac{2}{d}\right) n+R-\left(1-\frac{1}{d}\right)\left(\frac{n}{d}-R\right) T}\right\} .
$$

Proof. Let $S \in \mathbb{Z}$ and define

$$
m(S)=\operatorname{meas}\left\{\mathbf{v} \in \mathbb{T}^{n}:\left|f_{i}(\mathbf{v})\right|<q^{-S} \text { for } 1 \leqslant i \leqslant R\right\} .
$$

We claim that

$$
m(S)= \begin{cases}1 & \text { if } S<d  \tag{3.19}\\ q^{(d-1-S) n} N\left(t^{S+1-d}\right) & \text { if } S \geqslant d\end{cases}
$$

where

$$
N\left(\pi^{e}\right)=\#\left\{\mathbf{z} \in \mathbb{F}_{q}[t]^{n}:|\mathbf{z}|<\left|\pi^{e}\right|, f_{i}(\mathbf{z}) \equiv 0 \bmod \pi^{e} \text { for } 1 \leqslant i \leqslant R\right\},
$$

for any prime $\pi \in \mathbb{F}_{q}[t]$.
Taking (3.19) on faith for the moment, let us see how it suffices to complete the proof of the lemma. Taking the trivial upper bound $\left|S_{\infty}(\gamma)\right| \leqslant 1$, we see that $|\mathscr{I}(T)| \leqslant q^{R(T+1)}$. This is less than or equal to $q^{n-\left(1-\frac{1}{d}\right)\left(\frac{n}{d}-R\right) T}$ if $T \leqslant d-1$. Assume henceforth that $T \geqslant d$. It follows from (2.1) and (3.19) that

$$
\begin{aligned}
\mathscr{I}(T) & =q^{R(T+1)}\left(m(T+1)-q^{-R} m(T)\right) \\
& =q^{R(T+1)+(d-2-T) n}\left(N\left(t^{T+2-d}\right)-q^{n-R} N\left(t^{T+1-d}\right)\right)
\end{aligned}
$$

Taking $\pi=t$ and $e=T+2-d$, it now follows from (3.6) and Lemma 3.2 that

$$
\begin{aligned}
N\left(t^{T+2-d}\right)-q^{n-R} N\left(t^{T+1-d}\right) & =q^{(T+2-d)(n-R)} A\left(t^{T+2-d}\right) \\
& \ll q^{(T+2-d)(n-R)} \cdot q^{-\left(1-\frac{1}{d}\right)\left(\frac{n}{d}-R\right)(T+2-d)} .
\end{aligned}
$$

Hence

$$
\begin{aligned}
\mathscr{I}(T) & \ll q^{R(T+1)+(d-2-T) n} \cdot q^{(T+2-d)(n-R)} \cdot q^{-\left(1-\frac{1}{d}\right)\left(\frac{n}{d}-R\right)(T+2-d)} \\
& =q^{(d-1) R-\left(1-\frac{1}{d}\right)\left(\frac{n}{d}-R\right)(T+2-d)} \\
& =q^{g-\left(1-\frac{1}{d}\right)\left(\frac{n}{d}-R\right) T},
\end{aligned}
$$

where

$$
g=(d-1) R+(d-2)\left(1-\frac{1}{d}\right)\left(\frac{n}{d}-R\right) \leqslant\left(1-\frac{2}{d}\right) n+R,
$$

on taking $1-\frac{1}{d} \leqslant 1$.
We now turn to the proof of (3.19). If $S<d$ then the condition $\left|f_{i}(\mathbf{v})\right|<q^{-S}$ is vacuous for $1 \leqslant i \leqslant R$. In this case $m(S)=1$. Suppose next that $S \geqslant d$. Then

$$
\begin{aligned}
m(S) & =\int_{\mid \mathbf{w} \in \mathbb{T}^{n}} \#\left\{\left(\mathbf{u}_{1}, \ldots, \mathbf{u}_{S}\right) \in \mathbb{F}_{q}^{n S}:\left|f_{i}\left(t^{-S} \mathbf{u}_{1}+\cdots+t^{-1} \mathbf{u}_{S}+\mathbf{w}\right)\right|<q^{-S}\right\} \mathrm{d} \mathbf{w} \\
& =q^{-S n} \#\left\{\left(\mathbf{u}_{1}, \ldots, \mathbf{u}_{S}\right) \in \mathbb{F}_{q}^{n S}:\left|f_{i}\left(t^{-S} \mathbf{u}_{1}+\cdots+t^{-1} \mathbf{u}_{S}\right)\right|<q^{-S}\right\}
\end{aligned}
$$

since the integrand doesn't depend on $\mathbf{w}$. Let $\mathbf{u}=t^{-S} \mathbf{u}_{1}+\cdots+t^{-1} \mathbf{u}_{S}$ be one of the vectors appearing in the latter cardinality and suppose that

$$
f_{i}(\mathbf{u})=A_{0}+A_{1} t^{-1}+\cdots+A_{N} t^{-N}
$$

for some $A_{0}, \ldots, A_{N} \in \mathbb{F}_{q}$ and $N \geqslant 0$. Then the condition $\left|f_{i}(\mathbf{u})\right|<q^{-S}$ is equivalent to demanding that $A_{0}=\cdots=A_{S}=0$. We now consider the effect of writing $t^{-1}=\tau$ and working over $\mathbb{F}_{q}[\tau]$. Then $\mathbf{u}$ becomes $\mathbf{u}^{\prime}=\tau^{S} \mathbf{u}_{1}+\cdots+\tau \mathbf{u}_{S}$ and $f_{i}(\mathbf{u})$ becomes $f_{i}\left(\mathbf{u}^{\prime}\right)=A_{0}+A_{1} \tau+\cdots+A_{N} \tau^{N}$, with the condition $A_{0}=\cdots=A_{S}=0$ now being equivalent to $f_{i}\left(\mathbf{u}^{\prime}\right) \equiv 0 \bmod \tau^{S+1}$. Hence, on recalling that $S \geqslant d$, it follows that

$$
\begin{aligned}
m(S) & =q^{-S n} \#\left\{\mathbf{u} \in \mathbb{F}_{q}[\tau]^{n}:|\mathbf{u}|<q^{S}, f_{i}(\tau \mathbf{u}) \equiv 0 \bmod \tau^{S+1} \text { for } 1 \leqslant i \leqslant R\right\} \\
& =q^{-S n} \#\left\{\mathbf{u} \in \mathbb{F}_{q}[\tau]^{n}:|\mathbf{u}|<q^{S}, f_{i}(\mathbf{u}) \equiv 0 \bmod \tau^{S+1-d} \text { for } 1 \leqslant i \leqslant R\right\}
\end{aligned}
$$

Note that $S>S+1-d$, since $d \geqslant 2$. Breaking the $\mathbf{u}$ into residue classes modulo $\tau^{S+1-d}$, the claim (3.19) easily follows, which thereby completes the proof of the lemma.

We define the truncated singular integral

$$
\begin{equation*}
\mathfrak{I}(B)=\int_{|\gamma|<q^{B}} S_{\infty}(\boldsymbol{\gamma}) \mathrm{d} \boldsymbol{\gamma} \tag{3.20}
\end{equation*}
$$

for any $B \in \mathbb{Z}_{\geqslant 0}$, together with the completed singular integral

$$
\begin{equation*}
\mathfrak{I}=\int_{K_{\infty}^{R}} S_{\infty}(\gamma) \mathrm{d} \boldsymbol{\gamma} \tag{3.21}
\end{equation*}
$$

We are now ready to compare these two quantities, as summarised in the following result.

Lemma 3.8. Let $d \geqslant 2$ and assume that $n \geqslant d R$. Let $B \geqslant 0$. Then

$$
|\mathfrak{I}-\Im(B)| \ll q^{n-\left(1-\frac{1}{d}\right)\left(\frac{n}{d}-R\right) B} .
$$

Moreover, $\mathfrak{I}$ is absolutely convergent and satisfies $0<\mathfrak{I} \ll q^{n}$. The implied constants in these estimates depend only on $d, n, R$.

Proof. We clearly have $\left(1-\frac{2}{d}\right) n+R \leqslant n$ if $n \geqslant d R$. It follows from Lemma 3.7 that

$$
|\mathfrak{I}-\Im(B)| \leqslant \sum_{j \geqslant B}|\mathscr{I}(j)| \ll q^{n-\left(1-\frac{1}{d}\right)\left(\frac{n}{d}-R\right) B}
$$

as claimed in the first part of the lemma. The second part follows on taking

$$
|\mathfrak{I}| \leqslant \sum_{j \geqslant 0}|\mathscr{I}(j)|
$$

and reapplying Lemma 3.7. Finally, the last part follows from [20, Lemma 4.8.3], using the fact that $K_{\infty}$ is a $C_{1}$ field.

## 4. Application of the circle method

Throughout this section, let $\mathbb{F}_{q}$ be a finite field of characteristic $>d$ and let $f_{1}, \ldots, f_{R} \in \mathbb{F}_{q}\left[x_{1}, \ldots, x_{n}\right]$ be forms of degree $d$, cutting out a smooth complete intersection in $\mathbb{P}^{n-1}$. In particular it follows from Remark 2.3 that $\sigma \leqslant R-1$, in the notation of (2.2).

Let $m \in \mathbb{F}_{q}[t] \backslash\{0\}$ and $\mathbf{b} \in \mathbb{F}_{q}[t]^{n}$ with $\operatorname{deg} b_{i}<\operatorname{deg} m$ and $\operatorname{gcd}(\mathbf{b}, m)=1$. Our primary goal in this section is to adapt the work of Rydin Myerson [24] to produce an estimate for the quantity

$$
N(\mathbf{f} ; P, m, \mathbf{b})=\#\left\{\mathbf{g} \in \mathbb{F}_{q}[t]^{n}: \begin{array}{l}
|\mathbf{g}|<q^{P+\operatorname{deg} m} \text { and } \mathbf{g} \equiv \mathbf{b} \bmod m  \tag{4.1}\\
f_{1}(\mathbf{g})=\cdots=f_{R}(\mathbf{g})=0
\end{array}\right\}
$$

for any $P \in \mathbb{N}$. It will be crucial to have an estimate that depends explicitly on $P$ and on $q$.
4.1. Fourier analytic interpretation. Recall the definition (3.1) of

$$
\mathbf{F}(\mathbf{x})=\mathbf{f}(m \mathbf{x}+\mathbf{b})
$$

where we write $\mathbf{f}=\left(f_{1}, \ldots, f_{R}\right)$ for the vector of polynomials, and similarly for $\mathbf{F}$. Letting

$$
\begin{equation*}
S(\boldsymbol{\alpha} ; P)=S(\boldsymbol{\alpha} ; P, m, \mathbf{b})=\sum_{|\mathbf{x}|<q^{P}} \psi(\boldsymbol{\alpha} \cdot \mathbf{F}(\mathbf{x})), \tag{4.2}
\end{equation*}
$$

for $\boldsymbol{\alpha} \in \mathbb{T}^{R}$ and $P \in \mathbb{N}$, we may now write

$$
\begin{equation*}
N(\mathbf{f} ; P, m, \mathbf{b})=\int_{\mathbb{T}^{R}} S(\boldsymbol{\alpha} ; P) \mathrm{d} \boldsymbol{\alpha} \tag{4.3}
\end{equation*}
$$

Let $\mathscr{C}$ be an arbitrary real parameter that satisfies $\mathscr{C}>d R$. (We will make an acceptable choice of $\mathscr{C}$ in (5.1).) We will need to work under the following assumption. Given $\boldsymbol{\alpha} \in K_{\infty}^{R}$, we put $\|\boldsymbol{\alpha}\|=|\{\boldsymbol{\alpha}\}|$, where $\{\boldsymbol{\alpha}\} \in \mathbb{T}^{R}$ is the fractional part of $\boldsymbol{\alpha}$.

Hypothesis 4.1. There exists $C \geqslant 1$ such that

$$
\min \left\{\frac{|S(\boldsymbol{\alpha} ; P)|}{q^{n P}}, \frac{|S(\boldsymbol{\alpha}+\boldsymbol{\beta} ; P)|}{q^{n P}}\right\} \leqslant C \max \left\{q^{-d P+d-1}\left\|m^{d} \boldsymbol{\beta}\right\|^{-1},\left\|m^{d} \boldsymbol{\beta}\right\|^{\frac{1}{d-1}} q^{-1}\right\}^{\mathscr{C}},
$$

for any $\boldsymbol{\alpha}, \boldsymbol{\beta} \in K_{\infty}^{R}$ and any $P \in \mathbb{N}$.
The following result is a function field analogue of [24, Prop. 2.1], with the added feature that the dependence on $q$ is made precise.

Theorem 4.2. Let $f_{1}, \ldots, f_{R} \in \mathbb{F}_{q}\left[x_{1}, \ldots, x_{n}\right]$ be forms of degree $d \geqslant 2$. Let $\mathscr{C}>d R$ and suppose that Hypothesis 4.1 holds for $C \geqslant 1$. Assume that $P>R(d-1)+d \operatorname{deg} m$ and $\left(1-\frac{1}{d}\right)\left(\frac{n}{d}-R\right)>1$. Then

$$
N(\mathbf{f} ; P, m, \mathbf{b})=\mathfrak{S} \mathfrak{I} q^{(n-d R) P-d R \operatorname{deg} m}+O_{\varepsilon}(E),
$$

for any $\varepsilon>0$, with

$$
\begin{gathered}
E=q^{(n-\mathscr{C}) P}+q^{1+d R(1+\operatorname{deg} m)+\delta_{0}(1+d \operatorname{deg} m)\left(1-\frac{d R}{\mathscr{G}}\right)+\left(n-d R-\delta_{1}\right) P} \\
+q^{\left(n-d R-\delta_{2}+\varepsilon\right) P-(d-1) R \operatorname{deg} m+n+\delta_{2}(1+d \operatorname{deg} m+R(d-1))}
\end{gathered}
$$

and where

$$
\begin{align*}
& \delta_{0}=\frac{n-\sigma_{\mathbf{f}}}{(d-1) 2^{d-1} R}  \tag{4.4}\\
& \delta_{1}=\delta_{0}\left(1-\frac{d R}{\mathscr{C}}\right) \tag{4.5}
\end{align*}
$$

and $\delta_{2}=\left(1-\frac{1}{d}\right)\left(\frac{n}{d}-R\right)-1$. Moreover, the singular series $\mathfrak{S}$ is defined in (3.3) and the singular integral $\mathfrak{I}$ in (3.21). Finally, the implicit constant depends only on $d, n, R, \mathscr{C}, C$ and $\varepsilon$.

For each integer $J$ in the range $0 \leqslant J \leqslant P$, define

$$
\begin{equation*}
\mathfrak{M}(J)=\mathfrak{M}_{m}(J ; P)=\bigcup_{\substack{g \in \mathbb{F}_{q}[t] \text { monic } \\ 0<|g| \leqslant q^{J}}} \bigcup_{\substack{\left|\in \in \mathbb{F}_{q}[t] R\\\right| \operatorname{acc}<\left|\underline{l n} m^{d}\right| \\ \operatorname{gcd}(\mathbf{a}, g)=1}}\left\{\boldsymbol{\alpha} \in \mathbb{T}^{R}:\left|\boldsymbol{\alpha}-\frac{\mathbf{a}}{g m^{d}}\right|<\frac{q^{J-d P+d}}{\left|g m^{d}\right|}\right\} \tag{4.6}
\end{equation*}
$$

to be the major arcs of level $J$. The corresponding minor arcs are defined to be

$$
\mathfrak{m}(J)=\mathbb{T}^{R} \backslash \mathfrak{M}(J)
$$

With this notation we may break the integral (4.3) into major and minor arcs, obtaining

$$
\begin{equation*}
N(\mathbf{f} ; P, m, \mathbf{b})=\int_{\mathfrak{M}(\Delta(J))} S(\boldsymbol{\alpha} ; P) \mathrm{d} \boldsymbol{\alpha}+\int_{\mathfrak{m}(\Delta(J))} S(\boldsymbol{\alpha} ; P) \mathrm{d} \boldsymbol{\alpha} . \tag{4.7}
\end{equation*}
$$

We claim that the major arcs of level $J$ are non-overlapping if

$$
\begin{equation*}
J \leqslant \frac{d(P-1)}{2} \tag{4.8}
\end{equation*}
$$

To see this, if we have overlapping arcs associated to distinct $\frac{\mathrm{a}}{g m^{d}}$ and $\frac{\mathrm{a}^{\prime}}{g^{\prime} m^{d}}$, then we would be able to deduce that

$$
\frac{1}{|m|^{d}\left|g g^{\prime}\right|} \leqslant\left|\frac{\mathbf{a}}{g m^{d}}-\frac{\mathbf{a}^{\prime}}{g^{\prime} m^{d}}\right|<\frac{q^{J-d P+d}}{|m|^{d} \min \left\{|g|,\left|g^{\prime}\right|\right\}}
$$

by the ultrametric inequality. Since $|g|,\left|g^{\prime}\right| \leqslant q^{J}$, this leads to a contradiction under the inequality (4.8).

The first step in the treatment of the minor arcs will be the following result, which is a function field analogue of [24, Lemma 2.2].

Lemma 4.3. Let $\mathfrak{B} \subset \mathbb{T}^{R}$ be a measurable subset. Let $\mathscr{C}>d R$ and assume that Hypothesis 4.1 holds with $C \geqslant 1$. Suppose further that

$$
\sup _{\boldsymbol{\alpha} \in \mathfrak{B}}|S(\boldsymbol{\alpha} ; P)| \leqslant C_{0} q^{(n-\delta) P},
$$

for some $\delta>0$ and $C_{0} \geqslant 1$. Then

$$
\int_{\mathfrak{B}}|S(\boldsymbol{\alpha} ; P)| \mathrm{d} \boldsymbol{\alpha} \ll C_{0} q^{(n-\mathscr{C}) P}+C_{0} q^{1+d R(1+\operatorname{deg} m)+\left(n-d R-\delta\left(1-\frac{d R}{6}\right)\right) P},
$$

where the implicit constant depends only on $C, \mathscr{C}, d$ and $R$.
If $\delta$ is such that $\delta P \geqslant \mathscr{C}\left(1-\frac{d \operatorname{deg} m}{d-1}\right)$, then

$$
\int_{\mathfrak{B}}|S(\boldsymbol{\alpha} ; P)| \mathrm{d} \boldsymbol{\alpha} \ll C_{0} q^{(n-\mathscr{C}) P}+C_{0} q^{1+d R \operatorname{deg} m+\left(n-d R-\delta\left(1-\frac{d R}{\mathscr{G}}\right)\right) P} .
$$

Proof. Throughout this proof we allow any implied constant to depend on $C, \mathscr{C}, d$ and $R$. For each $j \in \mathbb{Z}$, we set $D(j)=\left\{\boldsymbol{\alpha} \in \mathfrak{B}:|S(\boldsymbol{\alpha} ; P)|>q^{j}\right\}$ and we put
$L(j)=$ meas $D(j)$. Note that $L(j) \leqslant 1$, since $\mathbb{T}^{R}$ has measure 1 . Then, for any integers $J \leqslant K$, we have

$$
\begin{aligned}
& \int_{\mathfrak{B}}|S(\boldsymbol{\alpha} ; P)| \mathrm{d} \boldsymbol{\alpha}= \int_{\mathfrak{B} \backslash D(J)}|S(\boldsymbol{\alpha} ; P)| \mathrm{d} \boldsymbol{\alpha}+\sum_{j=J}^{K-1} \int_{D(j) \backslash D(j+1)}|S(\boldsymbol{\alpha} ; P)| \mathrm{d} \boldsymbol{\alpha} \\
&+\int_{D(K)}|S(\boldsymbol{\alpha} ; P)| \mathrm{d} \boldsymbol{\alpha} \\
& \leqslant q^{J}+\sum_{j=J}^{K-1} L(j) q^{j+1}+L(K) \sup _{\boldsymbol{\alpha} \in \mathfrak{B}}|S(\boldsymbol{\alpha} ; P)| \\
& \leqslant q^{J}+\sum_{j=J}^{K-1} L(j) q^{j+1}+C_{0} q^{(n-\delta) P} L(K),
\end{aligned}
$$

by the hypothesis in the lemma. We shall take

$$
J=n P-\lceil\mathscr{C} P\rceil \quad \text { and } \quad K=n P-\lceil\delta P\rceil
$$

If $\delta P \geqslant \mathscr{C} P$ then the upper bound in the lemma is a trivial consequence of the hypothesis of the lemma, since $\mathfrak{B}$ has measure at most 1 . Hence we may proceed under the assumption that $J \leqslant K$.

We note that $q^{J} \leqslant q^{(n-\mathscr{C}) P}$ and so the first term is satisfactory. The second and third term combine to give

$$
\sum_{j=J}^{K-1} L(j) q^{j+1}+C_{0} q^{(n-\delta) P} L(K) \leqslant C_{0} \sum_{j=J}^{(n-\delta) P} L(j) q^{j+1}
$$

In particular, we have $j \leqslant n P-1$ in this sum. We claim that

$$
L(j) \ll \begin{cases}q^{d R\left(\operatorname{deg} m-P-\frac{(j-n P)}{\mathscr{C}}\right)} & \text { if } j \leqslant n P+\mathscr{C}\left(\frac{d \operatorname{deg} m}{d-1}-1\right),  \tag{4.9}\\ q^{d R\left(1-P-\frac{j-n P}{d \mathscr{C}}\right)} & \text { if } j>n P+\mathscr{C}\left(\frac{d \operatorname{deg} m}{d-1}-1\right),\end{cases}
$$

noting that the second case only occurs when $\operatorname{deg} m=0$. Taking this on faith for the moment, let us see how it suffices to complete the proof of the lemma.

When $\delta P \geqslant \mathscr{C}\left(1-\frac{d \operatorname{deg} m}{d-1}\right)$, we are always in the first case of (4.9) and we deduce that

$$
\begin{aligned}
\sum_{j=J}^{(n-\delta) P} L(j) q^{j+1} & \ll q^{d R(\operatorname{deg} m-P)} \sum_{j=J}^{(n-\delta) P} q^{j+1-\frac{(j-n P) d R}{6}} \\
& =q^{1+d R(\operatorname{deg} m-P)+\frac{d n R P}{\mathscr{}}} \sum_{j=J}^{(n-\delta) P} q^{j\left(1-\frac{d R}{6}\right)} \\
& \ll q^{1+d R(\operatorname{deg} m-P)+\frac{d n R P}{6}} q^{(n P-\delta P)\left(1-\frac{d R}{6}\right)} \\
& \leqslant q^{1+d R \operatorname{deg} m+(n-d R) P-\delta\left(1-\frac{d R}{6}\right) P},
\end{aligned}
$$

since $\mathscr{C}>d R$, by assumption. This is satisfactory for the second part of the lemma. Alternatively, when the assumption $\delta P \geqslant \mathscr{C}\left(1-\frac{d \operatorname{deg} m}{d-1}\right)$ is not made, we can combine the estimates in (4.9) to get the general upper bound $L(j) \ll q^{d R\left(1+\operatorname{deg} m-P-\frac{(j-n P)}{6}\right)}$, whence

$$
\begin{aligned}
\sum_{j=J}^{(n-\delta) P} L(j) q^{j+1} & \ll q^{d R(1+\operatorname{deg} m-P)} \sum_{j=J}^{(n-\delta) P} q^{j+1-\frac{(j-n P) d R}{\mathscr{C}}} \\
& \ll q^{1+d R(1+\operatorname{deg} m-P)+\frac{d n R P}{\mathscr{G}}} q^{(n P-\delta P)\left(1-\frac{d R}{\mathscr{G}}\right)} \\
& \leqslant q^{1+d R(1+\operatorname{deg} m)+(n-d R) P-\delta\left(1-\frac{d R}{6}\right) P},
\end{aligned}
$$

which completes the claim in the first part of the lemma.
It remains to establish (4.9) for any $j \leqslant n P-1$. For any $\boldsymbol{\alpha}, \boldsymbol{\alpha}+\boldsymbol{\beta} \in D(j)$, it follows from Hypothesis 4.1 that

$$
\left\|m^{d} \boldsymbol{\beta}\right\|<C^{1 / \mathscr{C}} q^{-d P+d-1-\frac{j-n P}{6}} \quad \text { or } \quad C^{-(d-1) / \mathscr{C}} q^{(j-n P) \frac{d-1}{6}+d-1}<\left\|m^{d} \boldsymbol{\beta}\right\| .
$$

We set

$$
\begin{aligned}
& r_{1}(j)=-d P+d-1-d \operatorname{deg} m-\frac{j-n P}{\mathscr{C}}+\frac{\log _{q} C}{\mathscr{C}} \\
& r_{2}(j)=(j-n P) \frac{d-1}{\mathscr{C}}-d \operatorname{deg} m+d-1-\frac{(d-1) \log _{q} C}{\mathscr{C}},
\end{aligned}
$$

in which notation the above conclusion can be written as

$$
\begin{equation*}
\left\|m^{d} \boldsymbol{\beta}\right\|<q^{\min \left\{0, r_{1}(j)+d \operatorname{deg} m\right\}} \quad \text { or } \quad q^{r_{2}(j)+d \operatorname{deg} m}<\left\|m^{d} \boldsymbol{\beta}\right\| . \tag{4.10}
\end{equation*}
$$

Recall the definition of the boxes from Section 2.4. As explained there, since $\mathfrak{B} \subset \mathbb{T}^{R}$, we can cover $D(j)$ by at most

$$
\max \left\{1, q^{R\left(-\left\lceil r_{2}(j)\right\rceil\right)}\right\} \leqslant \frac{1}{q^{R \min \left\{r_{2}(j), 0\right\}}}
$$

boxes of the form $B_{\left\lceil r_{2}(j)\right\rceil}\left(\mathbf{z}^{\prime}\right)$. Suppose that we are given any such box, with $\mathbf{z}^{\prime} \in K_{\infty}^{R}$. Then we claim that

$$
\begin{equation*}
\operatorname{meas}\left(B_{\left\lceil r_{2}(j)\right\rceil}\left(\mathbf{z}^{\prime}\right) \cap D(j)\right) \leqslant q^{R\left(r_{1}(j)+d \operatorname{deg} m\right)}, \tag{4.11}
\end{equation*}
$$

from which it follows that

$$
L(j) \ll q^{R\left(r_{1}(j)+d \operatorname{deg} m-\min \left\{0, r_{2}(j)\right\}\right)}
$$

Suppose first that $j \leqslant n P+\mathscr{C}\left(\frac{d \operatorname{deg} m}{d-1}-1\right)$. Then $\min \left\{0, r_{2}(j)\right\}=r_{2}(j)$ and it follows that $L(j) \ll q^{R\left(d \operatorname{deg} m-d P-\frac{(j-n P) d}{6}\right) \text {, as claimed. In the opposite case we have }}$ $\min \left\{0, r_{2}(j)\right\} \geqslant-\frac{(d-1) \log _{q} C}{\mathscr{C}}$ and it follows that

$$
L(j) \ll q^{d R(1-P)-\frac{(j-n P) R}{6}} .
$$

The bound claimed in (4.9) is now an easy consequence.

It remains to prove (4.11). Clearly there is nothing to prove if the intersection is empty. Thus, let us fix $\boldsymbol{\alpha} \in B_{\left\lceil r_{2}(j)\right\rceil}\left(\mathbf{z}^{\prime}\right) \cap D(j)$ and we suppose that $\boldsymbol{\alpha}+\boldsymbol{\beta} \in$ $B_{\left\lceil r_{2}(j)\right\rceil}\left(\mathbf{z}^{\prime}\right) \cap D(j)$, for some $\boldsymbol{\beta} \in \mathbb{T}^{R}$. Then

$$
\left\|m^{d} \boldsymbol{\beta}\right\|=\left\|m^{d}\left(\boldsymbol{\alpha}+\boldsymbol{\beta}-\mathbf{z}^{\prime}\right)-m^{d}\left(\boldsymbol{\alpha}-\mathbf{z}^{\prime}\right)\right\|<q^{\left[r_{2}(j)\right\rceil+d \operatorname{deg} m}
$$

Therefore, it follows from (4.10) that $\left\|m^{d} \boldsymbol{\beta}\right\|<q^{\min \left\{0, r_{1}(j)+d \operatorname{deg} m\right\}}$. The proof of (4.11) now follows on appealing to Lemma 2.2.
4.2. Treatment of the minor arcs. Recall from (2.5) the definition of the multilinear form associated to each form $f_{k}, 1 \leqslant k \leqslant R$. Then the multilinear form associated to $F_{k}$ is $m^{d} \Psi_{i}^{(k)}\left(\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(d-1)}\right)$, for $1 \leqslant k \leqslant R$. Recall the definition of $N^{(v)}(J ; \boldsymbol{\beta})$ from (2.15). The following result is a version of [20, Lemma 4.3.4] in which the implied constant has been made explicit.

Lemma 4.4. Let $J, P \in \mathbb{N}$ such that $J \leqslant P$ and let $\boldsymbol{\alpha} \in K_{\infty}^{R}$. Then we have

$$
|S(\boldsymbol{\alpha} ; P)|^{2^{d-1}} \leqslant q^{n\left(2^{d-1} P-(d-1) J\right)} N^{(d-1)}\left(J ; m^{d} \boldsymbol{\alpha}\right)
$$

Proof. We begin with an application of [20, Cor. 4.3.2], which leads to the inequality

$$
|S(\boldsymbol{\alpha} ; P)|^{2^{d-1}} \leqslant q^{\left(2^{d-1}-d+1\right) n P} N^{(0)}\left(J ; m^{d} \boldsymbol{\alpha}\right),
$$

for any $\boldsymbol{\alpha} \in K_{\infty}^{R}$. Therefore, it suffices to prove

$$
\begin{equation*}
N^{(0)}\left(J ; m^{d} \boldsymbol{\alpha}\right) \leqslant q^{-n(J-P)(d-1)} N^{(d-1)}\left(J ; m^{d} \boldsymbol{\alpha}\right) \tag{4.12}
\end{equation*}
$$

which in turn follows by showing that

$$
N^{(v-1)}\left(J ; m^{d} \boldsymbol{\alpha}\right) \leqslant q^{-n(J-P)} N^{(v)}\left(J ; m^{d} \boldsymbol{\alpha}\right),
$$

for each $1 \leqslant v \leqslant d-1$.
This will be a straightforward consequence of the version of the shrinking lemma found in work of Browning and Sawin [6, Lemma 6.4]. To see this, fix $1 \leqslant v \leqslant d-1$ and $\mathbf{x}^{(i)} \in \mathbb{F}_{q}[t]^{n}$, for each $i \neq v$, satisfying

$$
\begin{equation*}
\left|\mathbf{x}^{(1)}\right|, \ldots,\left|\mathbf{x}^{(v-1)}\right|<q^{J} \quad \text { and } \quad\left|\mathbf{x}^{(v+1)}\right|, \ldots,\left|\mathbf{x}^{(d-1)}\right|<q^{P} \tag{4.13}
\end{equation*}
$$

Let

$$
L_{i}(\mathbf{x})=\sum_{k=1}^{R} \alpha^{(k)} m^{d} \Psi_{i}^{(k)}\left(\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(v-1)}, \mathbf{x}, \mathbf{x}^{(v+1)} \ldots, \mathbf{x}^{(d-1)}\right)
$$

for $1 \leqslant i \leqslant n$. We write

$$
M(b ; Z)=\#\left\{(\mathbf{x}, \mathbf{y}) \in \mathbb{F}_{q}[t]^{2 n}:|\mathbf{x}|<q^{b+Z},\left|L_{i}(\mathbf{x})+y_{i}\right|<q^{-b+Z}, 1 \leqslant i \leqslant n\right\}
$$

for any $b, Z \in \mathbb{Z}$. Now choose $b, Z_{1}, Z_{2} \in \mathbb{Z}$ such that

$$
\begin{equation*}
Z_{2} \leqslant 0, \quad b+Z_{2} \in \mathbb{Z}, \quad b-Z_{2} \in \mathbb{Z}_{>0} \quad \text { and } \quad Z_{2}-Z_{1} \in \mathbb{Z}_{\geqslant 0} \tag{4.14}
\end{equation*}
$$

Taking $a=b+Z_{2} \in \mathbb{Z}, c=b-Z_{2} \in \mathbb{Z}_{>0}$ and $s=Z_{2}-Z_{1} \in \mathbb{Z}_{\geqslant 0}$ in [6, Lemma 6.4], it follows that

$$
\frac{M\left(b ; Z_{1}\right)}{M\left(b ; Z_{2}\right)} \geqslant q^{n\left(Z_{1}-Z_{2}\right)}
$$

for each $\mathbf{x}^{(i)}$ with $i \neq v$, such that (4.13) holds. Setting

$$
b=P \frac{v+1}{2}-J \frac{v-1}{2}, \quad Z_{1}=-P \frac{v+1}{2}+J \frac{v+1}{2}
$$

and

$$
Z_{2}=-P \frac{v-1}{2}+J \frac{v-1}{2},
$$

we clearly have

$$
b+Z_{2}=P, \quad b-Z_{2}=P v-J(v-1) \quad \text { and } \quad Z_{2}-Z_{1}=P-J,
$$

so that (4.14) holds. Summing the above inequality over all $\mathbf{x}^{(i)}$ with $i \neq v$, we arrive at the inequality $N^{(v)}\left(J ; m^{d} \boldsymbol{\alpha}\right) \geqslant q^{n(J-P)} N^{(v-1)}\left(J ; m^{d} \boldsymbol{\alpha}\right)$, which is what we were supposed to prove.

Given a form $f \in \mathbb{F}_{q}[t]\left[x_{1}, \ldots, x_{n}\right]$ of degree $d$, we put

$$
\mathscr{M}_{f}=\left\{\left(\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(d-1)}\right) \in \mathbb{A}^{(d-1) n}: \Psi_{j}\left(\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(d-1)}\right)=0,1 \leqslant j \leqslant n\right\},
$$

where $\Psi_{j}$ are the multilinear forms associated to $f$, and we let

$$
\mathscr{M}_{f}(J)=\mathscr{M}_{f} \cap\left\{\left(\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(d-1)}\right) \in \mathbb{F}_{q}[t]^{(d-1) n}:\left|\mathbf{x}^{(1)}\right|, \ldots,\left|\mathbf{x}^{(d-1)}\right|<q^{J}\right\} .
$$

Let $\mathbf{h} \in \mathbb{F}_{q}[t]^{R}$. Since

$$
V_{\text {h.f }}=\mathscr{M}_{\text {h.f }} \cap\left\{\left(\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(d-1)}\right) \in \mathbb{A}^{(d-1) n}: \mathbf{x}^{(1)}=\cdots=\mathbf{x}^{(d-1)}\right\},
$$

in the notation of (2.3), it follows that

$$
\operatorname{dim} V_{\text {h.f }}+(d-2) n \geqslant \operatorname{dim} \mathscr{M}_{\text {h.f }},
$$

by the affine dimension theorem. Hence

$$
\begin{equation*}
\# \mathscr{M}_{\mathbf{h} . \mathbf{f}}(J) \ll q^{\left((d-2) n+\operatorname{dim} V_{\text {h.f }}\right) J}, \tag{4.15}
\end{equation*}
$$

where the implicit constant depends only on $d$ and $n$.
Lemma 4.5. Let $J, P \in \mathbb{N}$ such that $J \leqslant P$ and let $\boldsymbol{\alpha}=\left(\alpha^{(1)}, \ldots, \alpha^{(R)}\right) \in K_{\infty}^{R}$. Then one of the following two alternatives holds:
(i) We have

$$
S(\boldsymbol{\alpha} ; P) \ll q^{n P-\frac{\left(n-\sigma_{f}\right)}{2^{d-1} J}}
$$

where the implicit constant depends only on $d, n, R$, and $\sigma_{\mathbf{f}}$ is defined in (2.4).
(ii) There exist $g, a_{1}, \ldots, a_{R} \in \mathbb{F}_{q}[t]$, with $g$ monic, such that $\operatorname{gcd}(g, \mathbf{a})=1$,

$$
0<|g| \leqslant q^{R(d-1)(J-1)}
$$

and

$$
\left|g m^{d} \boldsymbol{\alpha}-\mathbf{a}\right|<q^{-d P+R(d-1) J-(R-1)(d-1)} .
$$

Proof. For $1 \leqslant j \leqslant n$, let $\mathbf{M}_{j}$ be the matrix whose columns are

$$
\left(\begin{array}{c}
\Psi_{j}^{(1)}\left(\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(d-1)}\right) \\
\vdots \\
\Psi_{j}^{(R)}\left(\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(d-1)}\right)
\end{array}\right)
$$

for each $\left(\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(d-1)}\right)$ counted by $N^{(d-1)}\left(J ; m^{d} \boldsymbol{\alpha}\right)$, in the notation of (2.15). We then put these matrices together and define $\mathbf{M}=\left(\mathbf{M}_{1}, \ldots, \mathbf{M}_{n}\right)$. We consider two cases depending on the rank of $\mathbf{M}$.

Suppose first that rank $\mathbf{M}=R$. Then there exists a non-singular $R \times R$ submatrix

$$
\mathbf{M}_{0}=\left(\Psi_{j \ell}^{(i)}\left(\mathbf{x}_{\ell}^{(1)}, \ldots, \mathbf{x}_{\ell}^{(d-1)}\right)\right)_{1 \leqslant i, \ell \leqslant R}
$$

By the definition of $N^{(d-1)}\left(J ; m^{d} \boldsymbol{\alpha}\right)$ there exist $s^{(\ell)} \in \mathbb{F}_{q}[t]$ and $\left|\gamma^{(\ell)}\right|<q^{-d P+(d-1) J}$ satisfying

$$
\begin{equation*}
\sum_{i=1}^{R} m^{d} \alpha^{(i)} \Psi_{j_{\ell}}^{(i)}\left(\mathbf{x}_{\ell}^{(1)}, \ldots, \mathbf{x}_{\ell}^{(d-1)}\right)=s^{(\ell)}+\gamma^{(\ell)} \tag{4.16}
\end{equation*}
$$

for each $1 \leqslant \ell \leqslant R$. Let $\mathbf{C}=\left(c_{i, \ell}\right)_{1 \leqslant i, \ell \leqslant R}$ be the cofactor matrix of $\mathbf{M}_{0}$, so that $\mathbf{C}$ has entries in $\mathbb{F}_{q}[t]$ and satisfies

$$
\mathbf{M}_{0} \mathbf{C}^{T}=\left(\operatorname{det} \mathbf{M}_{0}\right) \mathbf{I}_{R}=\mathbf{C M}_{0}^{T}
$$

where $\mathbf{I}_{R}$ is the $R \times R$ identity matrix. Any entry in $\mathbf{C}$ has absolute value at most $q^{(R-1)(d-1)(J-1)}$, since each entry of $\mathbf{M}_{0}$ has absolute value $\leqslant q^{(d-1)(J-1)}$. On multiplying both sides of (4.16) by $\mathbf{C}$ on the left, we obtain

$$
\left(\begin{array}{c}
\left(\operatorname{det} \mathbf{M}_{0}\right) m^{d} \alpha^{(1)}-\sum_{\ell=1}^{R} c_{1, \ell} s^{(\ell)} \\
\vdots \\
\left(\operatorname{det} \mathbf{M}_{0}\right) m^{d} \alpha^{(R)}-\sum_{\ell=1}^{R} c_{R, \ell} s^{(\ell)}
\end{array}\right)=\mathbf{C}\left(\begin{array}{c}
\gamma^{(1)} \\
\vdots \\
\gamma^{(R)}
\end{array}\right) .
$$

We therefore obtain

$$
\begin{aligned}
\left|\left(\operatorname{det} \mathbf{M}_{0}\right) m^{d} \alpha^{(i)}-\sum_{\ell=1}^{R} c_{i, \ell} s^{(\ell)}\right| & <q^{(R-1)(d-1)(J-1)} q^{-d P+(d-1) J} \\
& =q^{-d P+R(d-1) J-(R-1)(d-1)}
\end{aligned}
$$

for $1 \leqslant i \leqslant R$. Set

$$
g=\frac{\operatorname{det} \mathbf{M}_{0}}{D} \quad \text { and } \quad a_{i}=\frac{1}{D} \sum_{\ell=1}^{R} c_{i, \ell} s^{(\ell)}
$$

for $1 \leqslant i \leqslant R$, where

$$
D=\operatorname{gcd}\left(\operatorname{det} \mathbf{M}_{0}, \sum_{\ell=1}^{R} c_{1, \ell} s^{(\ell)}, \ldots, \sum_{\ell=1}^{R} c_{R, \ell} s^{(\ell)}\right)
$$

On dividing through by an element of $\mathbb{F}_{q}^{*}$ we can further assume that $g$ is monic. Noting that $0<\left|\operatorname{det} \mathbf{M}_{0}\right| \leqslant q^{R(d-1)(J-1)}$, we therefore establish alternative (ii) of the lemma.

Next we suppose that rank $\mathbf{M}<R$. Then there exists $\mathbf{h} \in \mathbb{F}_{q}[t]^{R} \backslash\{\mathbf{0}\}$ such that

$$
\sum_{i=1}^{R} h_{i} \Psi_{j}^{(i)}\left(\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(d-1)}\right)=0
$$

for $1 \leqslant j \leqslant n$ and every $\left(\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(d-1)}\right)$ counted by $N^{(d-1)}\left(J ; m^{d} \boldsymbol{\alpha}\right)$. Since

$$
\sum_{i=1}^{R} h_{i} \Psi_{j}^{(i)}\left(\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(d-1)}\right)=\Psi_{j}^{(\mathbf{h})}\left(\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(d-1)}\right)
$$

for $1 \leqslant j \leqslant n$, where $\Psi_{j}^{(\mathbf{h})}$ are the multilinear forms associated to the form h.f. it then follows from (4.15) that

$$
N^{(d-1)}\left(J ; m^{d} \boldsymbol{\alpha}\right) \leqslant \# \mathscr{M}_{\mathbf{h} . \mathbf{f}}(J) \ll q^{J\left((d-2) n+\operatorname{dim} V_{\mathbf{h} . \mathbf{f}}\right)} \leqslant q^{J\left((d-2) n+\sigma_{\mathbf{f}}\right)}
$$

where the implicit constant depends only on $d$ and $n$, and $\sigma_{\mathbf{f}}$ is given in (2.4). Thus, by Lemma 4.4. it follows that

$$
S(\boldsymbol{\alpha} ; P) \ll q^{n P-\frac{\left(n-\sigma_{f}\right)}{2^{d-1}} J}
$$

which is precisely the bound in alternative (i).
Let us set

$$
\begin{equation*}
\Delta(J)=R(d-1)(J-1) \tag{4.17}
\end{equation*}
$$

and recall the definition (4.4) of $\delta_{0}$. Since the set of $\boldsymbol{\alpha}$ satisfying alternative (ii) of Lemma 4.5 is contained in the set of major $\operatorname{arcs} \mathfrak{M}(\Delta(J))$, in the notation of (4.6), the following result is an immediate consequence of Lemma 4.5,

Corollary 4.6. Let $J \leqslant P$ be integers. Then

$$
\sup _{\alpha \in \mathfrak{m}(\Delta(J))}|S(\boldsymbol{\alpha} ; P)| \ll q^{n P-(d-1) R \delta_{0} J}
$$

where the implicit constant depends only on $d, n$ and $R$, and where $\delta_{0}$ and $\Delta(J)$ are given by (4.4) and (4.17), respectively.
4.3. A major arc approximation. Recall the notation (3.2) for $S_{g}(\mathbf{a})$ and (3.17) for $S_{\infty}(\boldsymbol{\beta})$. We proceed by recording the following result, which refines work of Lee [20, Lemma 4.6.2]. (To be precise, Lee's version operates under the slightly more stringent assumption that $|\boldsymbol{\beta}|<q^{-(d-1) P}|g|^{-1}|m|^{-d}$.)
Lemma 4.7. Let $P \in \mathbb{N}$, let $g \in \mathbb{F}_{q}[t]$ and $\mathbf{a} \in \mathbb{F}_{q}[t]^{R}$. Put $\boldsymbol{\alpha}=\mathbf{a} / g+\boldsymbol{\beta}$ and assume that $0<|g| \leqslant q^{P}$ and $|\boldsymbol{\beta}|<q^{-(d-1)(P-1)}|g|^{-1}|m|^{-d}$. Then

$$
S\left(\frac{\mathbf{a}}{g}+\boldsymbol{\beta} ; P\right)=q^{n P} S_{g}(\mathbf{a}) S_{\infty}\left(m^{d} t^{d P} \boldsymbol{\beta}\right)
$$

Proof. We begin by deducing from (4.2) that

$$
S\left(\frac{\mathbf{a}}{g}+\boldsymbol{\beta} ; P\right)=\sum_{|\mathbf{x}|<q^{P}} \psi\left(\left(\frac{\mathbf{a}}{g}+\boldsymbol{\beta}\right) \cdot \mathbf{f}(m \mathbf{x}+\mathbf{b})\right) .
$$

Let us write $\mathbf{x}=g \mathbf{y}+\mathbf{z}$, where $|\mathbf{y}|<q^{P}|g|^{-1}$ and $|\mathbf{z}|<|g|$. Then it follows that

$$
\begin{aligned}
S\left(\frac{\mathbf{a}}{g}+\boldsymbol{\beta} ; P\right) & =\sum_{|\mathbf{y}|<q^{P}|g|^{-1}} \sum_{|\mathbf{z}|<|g|} \psi\left(\left(\frac{\mathbf{a}}{g}+\boldsymbol{\beta}\right) \cdot \mathbf{f}(m(g \mathbf{y}+\mathbf{z})+\mathbf{b})\right) \\
& =\sum_{|\mathbf{z}|<|g|} \psi\left(\frac{\mathbf{a} \cdot \mathbf{f}(m \mathbf{z}+\mathbf{b})}{g}\right) \sum_{|\mathbf{y}|<q^{P}|g|^{-1}} \psi(\boldsymbol{\beta} \cdot \mathbf{f}(m(g \mathbf{y}+\mathbf{z})+\mathbf{b})) .
\end{aligned}
$$

We claim that $\psi(\boldsymbol{\beta} \cdot \mathbf{f}(m(g \mathbf{y}+\mathbf{z})+\mathbf{b}))=\psi(\boldsymbol{\beta} \cdot \mathbf{f}(m g \mathbf{y}+\mathbf{b}))$ in the inner sum, under the assumptions of the lemma, from which it will follow that

$$
\begin{equation*}
S\left(\frac{\mathbf{a}}{g}+\boldsymbol{\beta} ; P\right)=|g|^{n} S_{g}(\mathbf{a}) \sum_{|\mathbf{y}|<q^{P}|g|^{-1}} \psi(\boldsymbol{\beta} . \mathbf{f}(m g \mathbf{y}+\mathbf{b})) . \tag{4.18}
\end{equation*}
$$

To check the claim it will suffice to show that

$$
\begin{equation*}
\operatorname{ord}(\boldsymbol{\beta} \cdot(\mathbf{f}(m(g \mathbf{y}+\mathbf{z})+\mathbf{b})-\mathbf{f}(m g \mathbf{y}+\mathbf{b})))<-1 \tag{4.19}
\end{equation*}
$$

for any $|\mathbf{z}|<|g|$. Note that $|m g \mathbf{y}+\mathbf{b}| \leqslant q^{\operatorname{deg} m+P-1}$. Hence it follows from Taylor expansion that the left hand side of this expression is

$$
\begin{aligned}
\operatorname{ord}(\boldsymbol{\beta} . & (\mathbf{f}(m(g \mathbf{y}+\mathbf{z})+\mathbf{b})-\mathbf{f}(m g \mathbf{y}+\mathbf{b}))) \\
& \leqslant \operatorname{ord} \boldsymbol{\beta}+\max (\operatorname{ord}(m \mathbf{z})+(d-1) \operatorname{ord}(m g \mathbf{y}+\mathbf{b}), d \operatorname{ord}(m \mathbf{z})) \\
& \leqslant \operatorname{ord} \boldsymbol{\beta}+\max \left(\operatorname{deg} m^{d}+\operatorname{deg} g-1+(d-1)(P-1), \operatorname{deg} m^{d}+d(\operatorname{deg} g-1)\right) \\
& =\operatorname{ord} \boldsymbol{\beta}+(d-1)(P-1)+\operatorname{deg} g+\operatorname{deg} m^{d}-1 \\
& <-1
\end{aligned}
$$

as required. Now that we have verified (4.18), a further application of (4.19) reveals that

$$
\begin{aligned}
\sum_{|\mathbf{y}|<q^{P}|g|^{-1}} \psi(\boldsymbol{\beta} \cdot \mathbf{f}(m g \mathbf{y}+\mathbf{b})) & =|g|^{-n} \sum_{|\mathbf{z}|<|g||\mathbf{y}|<q^{P}|g|^{-1}} \sum_{\mid \boldsymbol{\beta} \cdot \mathbf{f}(m(g \mathbf{y}+\mathbf{z})+\mathbf{b}))} \\
& =|g|^{-n} \sum_{|\mathbf{x}|<q^{P}} \psi(\boldsymbol{\beta} \cdot \mathbf{f}(m \mathbf{x}+\mathbf{b})) .
\end{aligned}
$$

Hence

$$
\begin{equation*}
S\left(\frac{\mathbf{a}}{g}+\boldsymbol{\beta} ; P\right)=S_{g}(\mathbf{a}) S(\boldsymbol{\beta} ; P) \tag{4.20}
\end{equation*}
$$

Arguing as in (4.19), it is easy to check that

$$
\operatorname{ord}(\boldsymbol{\beta} \cdot(\mathbf{f}(m(\mathbf{x}+\boldsymbol{\sigma})+\mathbf{b})-\mathbf{f}(m \mathbf{x}+\mathbf{b})))<-1
$$

for any $|\mathbf{x}|<q^{P}$ and $\boldsymbol{\sigma} \in \mathbb{T}^{n}$. Thus $\psi(\boldsymbol{\beta} \cdot \mathbf{f}(m \mathbf{x}+\mathbf{b}))=\psi(\boldsymbol{\beta} \cdot \mathbf{f}(m \mathbf{v}+\mathbf{b}))$, for any $|\mathbf{x}-\mathbf{v}|<1$. It now follows that

$$
\begin{aligned}
S(\boldsymbol{\beta} ; P) & =\sum_{|\mathbf{x}|<q^{P}} \int_{|\mathbf{x}-\mathbf{v}|<1} \psi(\boldsymbol{\beta} \cdot \mathbf{f}(m \mathbf{x}+\mathbf{b})) \mathrm{d} \mathbf{v} \\
& =\sum_{|\mathbf{x}|<q^{P}} \int_{|\mathbf{x}-\mathbf{v}|<1} \psi(\boldsymbol{\beta} \cdot \mathbf{f}(m \mathbf{v}+\mathbf{b})) \mathrm{d} \mathbf{v} \\
& =\int_{|\mathbf{v}|<q^{P}} \psi(\boldsymbol{\beta} \cdot \mathbf{f}(m \mathbf{v}+\mathbf{b})) \mathrm{d} \mathbf{v} \\
& =q^{n P} \int_{\mathbb{T}^{n}} \psi\left(\boldsymbol{\beta} \cdot \mathbf{f}\left(t^{P} m \mathbf{v}+\mathbf{b}\right)\right) \mathrm{d} \mathbf{v}
\end{aligned}
$$

on making a change of variables. Next, we claim that

$$
\operatorname{ord}\left(t^{d P} \boldsymbol{\beta} \cdot\left(\mathbf{f}\left(m \mathbf{v}+\mathbf{b} t^{-P}\right)-\mathbf{f}(m \mathbf{v})\right)\right)<-1
$$

for any $\mathbf{v} \in \mathbb{T}^{n}$. Taking this on faith for the moment, this will imply that

$$
S(\boldsymbol{\beta} ; P)=q^{n P} \int_{\mathbb{T}^{n}} \psi\left(m^{d} t^{d P} \boldsymbol{\beta} \cdot \mathbf{f}(\mathbf{v})\right) \mathrm{d} \mathbf{v}=q^{n P} S_{\infty}\left(m^{d} t^{d P} \boldsymbol{\beta}\right)
$$

which will complete the proof of the lemma, on substituting back into (4.20). To check the claim, we note that the left hand side is

$$
\begin{aligned}
& \leqslant d P+\operatorname{ord} \boldsymbol{\beta}+\max \left(\operatorname{ord}\left(\mathbf{b} t^{-P}\right)+(d-1) \operatorname{ord}(m \mathbf{v}), d \operatorname{ord}\left(\mathbf{b} t^{-P}\right)\right) \\
& \leqslant d P+\operatorname{ord} \boldsymbol{\beta}+\max (\operatorname{deg} m-1-P+(d-1)(\operatorname{deg} m-1), d(\operatorname{deg} m-1-P)) \\
& =d P+\operatorname{ord} \boldsymbol{\beta}-P+d(\operatorname{deg} m-1) \\
& =\operatorname{ord} \boldsymbol{\beta}+(d-1)(P-1)+\operatorname{deg} m^{d}-1 \\
& <-1
\end{aligned}
$$

as required.
4.4. Deduction of Theorem 4.2, Let $\mathscr{C}>d R$. We now have everything in place to complete the proof of Theorem 4.2, which we tackle under the assumption that Hypothesis 4.1 holds and $P>R(d-1)+d \operatorname{deg} m$. Recall the notation for $\Delta(J)$ in (4.17) and the definition of the major arcs $\mathfrak{M}(\Delta(J))=\mathfrak{M}_{m}(\Delta(J) ; P)$ in (4.6). We require the major arcs to be non-overlapping, which in view of (4.8), means that we need

$$
\begin{equation*}
\Delta(J) \leqslant \frac{d(P-1)}{2} \tag{4.21}
\end{equation*}
$$

We shall apply Lemma 4.7 with $g m^{d}$ in place of $g$. We also need to choose $J$ in order that this lemma is applicable on the major arcs $\mathfrak{M}(\Delta(J))$. Thus we need $\left|g m^{d}\right| \leqslant q^{P}$ and $\left|{g m^{d}}^{d} \boldsymbol{\beta}\right|<q^{-(d-1)(P-1)}|m|^{-d}$; in other words, we need

$$
|g| \leqslant q^{P-d \operatorname{deg} m} \quad \text { and } \quad|g \boldsymbol{\beta}|<q^{-(d-1)(P-1)-2 d \operatorname{deg} m} .
$$

But $\boldsymbol{\alpha} \in \mathfrak{M}(\Delta(J))$ implies that $|g| \leqslant q^{\Delta(J)}$ and $|g \boldsymbol{\beta}|<q^{\Delta(J)-d P+d}|m|^{-d}$. Thus we need $\Delta(J) \leqslant P-d \operatorname{deg} m$ and $\Delta(J)-d P+d \leqslant-(d-1)(P-1)-d \operatorname{deg} m$. These
two inequalities are satisfied if $\Delta(J) \leqslant P-1-d \operatorname{deg} m$, which in the light of (4.17), we can achieve by setting

$$
\begin{equation*}
J=\left\lfloor\frac{P-1-d \operatorname{deg} m}{R(d-1)}\right\rfloor+1 \geqslant \frac{P-1-d \operatorname{deg} m}{R(d-1)} . \tag{4.22}
\end{equation*}
$$

This is also enough to ensure that (4.21) holds.
We break the integral into major and minor arcs, as in (4.7). For the minor arcs, we apply the first part of Lemma 4.3 with $\mathfrak{B}=\mathfrak{m}(\Delta(J))$ and $\delta=(d-1) R \delta_{0} J / P$, and where $C_{0}$ is the implicit constant from Corollary 4.6. Hence it follows that

$$
\int_{\mathfrak{m}(\Delta(J))}|S(\boldsymbol{\alpha} ; P)| \mathrm{d} \boldsymbol{\alpha} \ll q^{(n-\mathscr{C}) P}+q^{1+d R(1+\operatorname{deg} m)+(n-d R) P-(d-1) R \delta_{0} J\left(1-\frac{d R}{\mathscr{C}}\right)} .
$$

Recalling that $\mathscr{C}>d R$, together with the lower bound (4.22), we see that

$$
\begin{aligned}
& 1+d R(1+\operatorname{deg} m)+(n-d R) P-(d-1) R \delta_{0} J\left(1-\frac{d R}{\mathscr{C}}\right) \\
& \quad \leqslant 1+d R(1+\operatorname{deg} m)+\left(n-d R-\delta_{1}\right) P+\delta_{0}(1+d \operatorname{deg} m)\left(1-\frac{d R}{\mathscr{C}}\right)
\end{aligned}
$$

where $\delta_{1}$ is given by (4.5). Hence

$$
\begin{equation*}
\int_{\mathfrak{m}(\Delta(J))}|S(\boldsymbol{\alpha} ; P)| \mathrm{d} \boldsymbol{\alpha} \ll q^{(n-\mathscr{C}) P}+q^{1+d R(1+\operatorname{deg} m)+\delta_{0}(1+d \operatorname{deg} m)\left(1-\frac{d R}{\mathscr{G}}\right)+\left(n-d R-\delta_{1}\right) P} . \tag{4.23}
\end{equation*}
$$

Turning to the major arcs, it follows from Lemma 4.7 that

$$
\begin{aligned}
& \int_{\mathfrak{M}(\Delta(J))} S(\boldsymbol{\alpha} ; P) \mathrm{d} \boldsymbol{\alpha}=\sum_{\substack{\left.g \in \mathbb{F}_{q} \mid t\right] \\
|g| \leqslant q^{\Delta(J)} \\
\text { monic }}} \sum_{\substack{\left.\mathbf{a} \in \mathbb{F}_{q} \mid t\right)^{R} \\
|\mathbf{a}|<\left|g m^{d}\right| \\
\operatorname{gcd}(\mathbf{a}, g)=1}} \int_{|\boldsymbol{\beta}|<\frac{q^{\Delta(J)-d P+d}}{\left|g m^{d}\right|}} S\left(\frac{\mathbf{a}}{g m^{d}}+\boldsymbol{\beta} ; P\right) \mathrm{d} \boldsymbol{\beta} \\
& =q^{(n-d R) P-d R \operatorname{deg} m} \sum_{\substack{g \in \mathbb{F}_{q}[t] \text { monic } \\
|g| \leqslant q^{\Delta(J)}}} \sum_{\substack{\mathbf{a} \in \mathbb{F}_{q}\left[t R ^ { R } \\
\left|\mathbf{a}<\left|g m^{d}\right| \\
\operatorname{gcd}(\mathbf{a}, g)=1\right.\right.}} S_{g m^{d}}(\mathbf{a}) \mathfrak{I}(B),
\end{aligned}
$$

where $\Im(B)$ is given by (3.20), with $B=\Delta(J)+d-\operatorname{deg} g$. According to the first part of Lemma 3.8, we have

$$
\begin{aligned}
|\mathfrak{I}-\mathfrak{I}(B)| & \ll q^{n-\left(1-\frac{1}{d}\right)\left(\frac{n}{d}-R\right) B} \\
& \ll q^{\frac{n}{d}+d R-R-\left(1-\frac{1}{d}\right)\left(\frac{n}{d}-R\right) \Delta(J)}|g|^{\left(1-\frac{1}{d}\right)\left(\frac{n}{d}-R\right)} .
\end{aligned}
$$

Appealing to Lemma 3.4 and recalling the notation (3.16), this error term contributes

$$
\ll q^{(n-d R) P-d R \operatorname{deg} m+\frac{n}{d}+(d-1) R-\left(1-\frac{1}{d}\right)\left(\frac{n}{d}-R\right) \Delta(J)}|m|^{R} \sum_{T=0}^{\Delta(J)} q^{\left(1-\frac{1}{d}\right)\left(\frac{n}{d}-R\right) T} \mathscr{S}(T) .
$$

Next, it follows from Lemma 3.5 that this contribution is

$$
\begin{aligned}
& <_{\varepsilon} q^{(n-d R) P-(d-1) R \operatorname{deg} m+\frac{n}{d}+(d-1) R-\left(1-\frac{1}{d}\right)\left(\frac{n}{d}-R\right) \Delta(J)+P \varepsilon} \sum_{T=0}^{\Delta(J)} q^{T} \\
& <_{\varepsilon} q^{(n-d R) P-(d-1) R \operatorname{deg} m+\frac{n}{d}+(d-1) R-\left(\left(1-\frac{1}{d}\right)\left(\frac{n}{d}-R\right)-1\right) \Delta(J)+P \varepsilon},
\end{aligned}
$$

for any $\varepsilon>0$. Finally, we may apply Lemmas 3.6 and 3.8 to deduce that

$$
\mathfrak{S}(\Delta(J)) \mathfrak{I}=\mathfrak{S} \mathfrak{I}+O_{\varepsilon}\left(q^{n+R \operatorname{deg} m+(\Delta(J)+1)\left(1-\left(1-\frac{1}{d}\right)\left(\frac{n}{d}-R\right)\right)+P \varepsilon}\right)
$$

Let

$$
E_{\text {major }}=\left|\int_{\mathfrak{M}(\Delta(J))} S(\boldsymbol{\alpha} ; P) \mathrm{d} \boldsymbol{\alpha}-\mathfrak{S} \Im q^{(n-d R) P-d R \operatorname{deg} m}\right|,
$$

where $\mathfrak{S}$ and $\mathfrak{I}$ are given by (3.3) and (3.21), respectively. Then we have shown that

$$
\begin{aligned}
E_{\text {major }}<_{\varepsilon} & q^{(n-d R) P-(d-1) R \operatorname{deg} m+P \varepsilon}\left(q^{\frac{n}{d}+(d-1) R}+q^{n-\left(\left(1-\frac{1}{d}\right)\left(\frac{n}{d}-R\right)-1\right)}\right) \\
& \times q^{-\left(\left(1-\frac{1}{d}\right)\left(\frac{n}{d}-R\right)-1\right) \Delta(J)},
\end{aligned}
$$

for any $\varepsilon>0$, on recalling our assumption that $\left(1-\frac{1}{d}\right)\left(\frac{n}{d}-R\right)>1$. Under this assumption one easily checks that the middle term in brackets is at most $2 q^{n}$. Moreover, it follows from (4.17) and (4.22) that

$$
\Delta(J)=R(d-1)(J-1) \geqslant P-1-d \operatorname{deg} m-R(d-1)
$$

Note that $\Delta(J) \geqslant 0$ under the assumptions of Theorem 4.2. Hence we obtain

$$
E_{\text {major }} \ll \varepsilon q^{\left(n-d R-\delta_{2}+\varepsilon\right) P-(d-1) R \operatorname{deg} m+n+\delta_{2}(1+d \operatorname{deg} m+R(d-1))},
$$

where $\delta_{2}=\left(1-\frac{1}{d}\right)\left(\frac{n}{d}-R\right)-1$. Finally, we complete the proof of Theorem 4.2 on combining this with (4.23) in (4.7).

## 5. Arithmetic deductions

The main goal of this section is to prove Theorem 1.3. We begin by proving Hypothesis 4.1 for a suitable $C \geqslant 1$, for which we will need to assume that $q \geqslant(d-1)^{n}$. Recalling the definition of $N^{(d-1)}(J ; \boldsymbol{\beta})$ from (2.15), and the definition of the sum $S(\boldsymbol{\alpha} ; P)=S(\boldsymbol{\alpha} ; P, m, \mathbf{b})$ from (4.2), the first step is to prove the following result.

Lemma 5.1. Let $J, P \in \mathbb{N}$ such that $J \leqslant P$. Then

$$
\min \left\{\frac{|S(\boldsymbol{\alpha} ; P)|}{q^{n P}}, \frac{|S(\boldsymbol{\alpha}+\boldsymbol{\beta} ; P)|}{q^{n P}}\right\}^{2^{d}} \leqslant q^{-n(d-1) J} N^{(d-1)}\left(J ; m^{d} \boldsymbol{\beta}\right)
$$

for any $\boldsymbol{\alpha}, \boldsymbol{\beta} \in K_{\infty}^{R}$.

Proof. Let us write $S(\boldsymbol{\alpha} ; P)=S(\boldsymbol{\alpha})$ for simplicity. We start by noting that

$$
\begin{aligned}
& \min \{|S(\boldsymbol{\alpha})|,|S(\boldsymbol{\alpha}+\boldsymbol{\beta})|\}^{2} \\
& \leqslant|S(\boldsymbol{\alpha})||S(\boldsymbol{\alpha}+\boldsymbol{\beta})| \\
&=|S(\boldsymbol{\alpha}+\boldsymbol{\beta}) S(-\boldsymbol{\alpha})| \\
&=\sum_{|\mathbf{x}|<q^{P}} \sum_{|\mathbf{y}|<q^{P}} \psi((\boldsymbol{\alpha}+\boldsymbol{\beta}) \cdot \mathbf{f}(m \mathbf{x}+\mathbf{b})-\boldsymbol{\alpha} \cdot \mathbf{f}(m \mathbf{x}+m \mathbf{y}+\mathbf{b})) \\
& \leqslant \sum_{|\mathbf{y}|<q^{P}}\left|\widetilde{S}_{\boldsymbol{\alpha}, \mathbf{y}}(\boldsymbol{\beta})\right|
\end{aligned}
$$

where

$$
\widetilde{S}_{\alpha, \mathbf{y}}(\boldsymbol{\beta})=\sum_{|\mathbf{x}|<q^{P}} \psi\left(m^{d} \boldsymbol{\beta} \cdot \mathbf{f}(\mathbf{x})+G_{\boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{y}, m, \mathbf{b}}(\mathbf{x})\right)
$$

for a suitable polynomial $G_{\boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{y}, m, \mathbf{b}}$ of degree $<d$. Precisely the same argument used in [20, Cor. 4.3.2] now yields

$$
\left|\widetilde{S}_{\boldsymbol{\alpha}, \mathbf{y}}(\boldsymbol{\beta})\right|^{2^{d-1}} \leqslant q^{\left(2^{d-1}-d+1\right) n P} N^{(0)}\left(J ; m^{d} \boldsymbol{\beta}\right)
$$

where $N^{(0)}(J ; \boldsymbol{\beta})$ is defined in (2.15). (Note that the term $G_{\boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{y}, m, \mathbf{b}}$ gets eliminated during the Weyl differencing process, since it has degree $<d$.) An application of (4.12) implies

$$
\left|\widetilde{S}_{\boldsymbol{\alpha}, \mathbf{y}}(\boldsymbol{\beta})\right|^{2^{d-1}} \leqslant q^{n 2^{d-1} P-n(d-1) J} N^{(d-1)}\left(J ; m^{d} \boldsymbol{\beta}\right)
$$

With this estimate, we may deduce that

$$
\begin{aligned}
\min \{|S(\boldsymbol{\alpha})|,|S(\boldsymbol{\alpha}+\boldsymbol{\beta})|\}^{2} & \leqslant \sum_{|\mathbf{y}|<q^{P}} q^{n P-\frac{n(d-1)}{2^{d-1} J}} N^{(d-1)}\left(J ; m^{d} \boldsymbol{\beta}\right)^{\frac{1}{2^{d-1}}} \\
& =q^{2 n P-\frac{n(d-1)}{2^{d-1}} J} N^{(d-1)}\left(J ; m^{d} \boldsymbol{\beta}\right)^{\frac{1}{2^{d-1}}} .
\end{aligned}
$$

The result follows by rearranging this inequality.
With this lemma to hand, we may now invoke the estimates from Section 2.5 to prove Hypothesis 4.1. Let

$$
\begin{equation*}
\mathscr{C}=\frac{1}{2^{d+\nu(d)}(d-1)}\left(\left(1-\log _{q}(d-1)\right) n-R+1\right) \tag{5.1}
\end{equation*}
$$

where

$$
\nu(d)= \begin{cases}1 & \text { if } d=2  \tag{5.2}\\ 0 & \text { if } d \geqslant 3\end{cases}
$$

We shall prove that Hypothesis 4.1 holds with

$$
C=(d-1)^{\frac{n}{2^{d}}},
$$

under the assumptions $q \geqslant(d-1)^{n}$ and $\mathscr{C} \geqslant 1$. (Later we shall need to assume that $\mathscr{C}>d R$.) Let $P \in \mathbb{N}$ and let $\boldsymbol{\alpha}, \boldsymbol{\beta} \in K_{\infty}^{R}$. Let $C_{0}=(d-1)^{n}>1$ and write

$$
\gamma=C_{0}^{\frac{1}{2^{d}}} \min \left\{\frac{|S(\boldsymbol{\alpha} ; P)|}{q^{n P}}, \frac{|S(\boldsymbol{\alpha}+\boldsymbol{\beta} ; P)|}{q^{n P}}\right\}^{-1}
$$

Note that $\gamma>1$. Thus there exists $N \in \mathbb{Z}_{\geqslant 0}$ and $\varepsilon_{0} \in[0,1)$ such that

$$
\gamma^{1 / \mathscr{C}}=q^{N+\varepsilon_{0}} .
$$

We must now differentiate according to whether or not $N$ is positive.

The case $N \geqslant 1$. Suppose first that $N \geqslant 1$ and let $\Delta=\varepsilon_{0} \mathscr{C} / N \geqslant 0$. Then

$$
\gamma^{1 /(\mathscr{C}+\Delta)}=q^{N}
$$

and we define $\eta_{0}=N / P$. Suppose for the moment that $\eta_{0}>1$, so that $N>P$. Then

$$
\begin{aligned}
\min \left\{\frac{|S(\boldsymbol{\alpha} ; P)|}{q^{n P}}, \frac{|S(\boldsymbol{\alpha}+\boldsymbol{\beta} ; P)|}{\left.q^{n P}\right\}}\right. & =C_{0}^{\frac{1}{2 d}} \gamma^{-1} \\
& =C_{0}^{\frac{1}{2 d}} q^{-N(\mathscr{C}+\Delta)} \\
& \leqslant C_{0}^{\frac{1}{2 d}} q^{-P \mathscr{C}} .
\end{aligned}
$$

But $q^{-P} \leqslant \max \left\{q^{-d P+d-1}\left\|m^{d} \boldsymbol{\beta}\right\|^{-1},\left\|m^{d} \boldsymbol{\beta}\right\|^{\frac{1}{d-1}} q^{-1}\right\}$, as checked by considering the two cases $q^{-d P+d-1}\left\|m^{d} \boldsymbol{\beta}\right\|^{-1} \leqslant\left\|m^{d} \boldsymbol{\beta}\right\|^{\frac{1}{d-1}} q^{-1}$ and $q^{-d P+d-1}\left\|m^{d} \boldsymbol{\beta}\right\|^{-1}>\left\|m^{d} \boldsymbol{\beta}\right\|^{\frac{1}{d-1}} q^{-1}$ separately. In particular it follows that

$$
\min \left\{\frac{|S(\boldsymbol{\alpha} ; P)|}{q^{n P}}, \frac{|S(\boldsymbol{\alpha}+\boldsymbol{\beta} ; P)|}{q^{n P}}\right\} \leqslant C_{0}^{\frac{1}{2^{d}}} \max \left\{q^{-d P+d-1}\left\|m^{d} \boldsymbol{\beta}\right\|^{-1},\left\|m^{d} \boldsymbol{\beta}\right\|^{\frac{1}{d-1}} q^{-1}\right\}^{\mathscr{C}},
$$

which implies Hypothesis 4.1 with $C=C_{0}^{\frac{1}{2 d}}$.
We may therefore proceed under the assumption that $\eta_{0} P \in \mathbb{Z} \cap[1, P]$. In this case

$$
C_{0}^{\frac{1}{2 d \mathscr{C}}} \min \left\{\frac{|S(\boldsymbol{\alpha} ; P)|}{q^{n P}}, \frac{|S(\boldsymbol{\alpha}+\boldsymbol{\beta} ; P)|}{q^{n P}}\right\}^{-1 / \mathscr{C}}=\gamma^{1 / \mathscr{C}}=q^{N+\varepsilon_{0}}
$$

Since $N+\varepsilon_{0} \geqslant N=\eta_{0} P$, we obtain

$$
C_{0}^{-\frac{1}{2 d \mathscr{C}}} \min \left\{\frac{|S(\boldsymbol{\alpha} ; P)|}{q^{n P}}, \frac{|S(\boldsymbol{\alpha}+\boldsymbol{\beta} ; P)|}{q^{n P}}\right\}^{1 / \mathscr{C}} \leqslant q^{-\eta_{0} P}
$$

Suppose that alternative (i) holds in Lemma 2.7, with $J=\eta_{0} P$ and $\left\{m^{d} \boldsymbol{\beta}\right\}$. Then it immediately follows that Hypothesis 4.1 holds with $C=C_{0}^{\frac{1}{2 d}}$. If, on the other hand, alternative (ii) holds, then

$$
q^{\eta_{0} P-d P+d-1} \leqslant\left|\left\{m^{d} \boldsymbol{\beta}\right\}\right| \leqslant q^{-\eta_{0} P(d-1)+d-2}
$$

and $N^{(d-1)}\left(\eta_{0} P ;\left\{m^{d} \boldsymbol{\beta}\right\}\right)=N^{\text {aux }}\left(\eta_{0} P ;\left\{m^{d} \boldsymbol{\beta}\right\} t^{d P-\eta_{0} P}\right)$, in the notation of (2.6) and (2.15). We have

$$
\begin{aligned}
q^{n(d-1) \eta_{0} P} \gamma^{-2^{d}} & =C_{0}^{-1} q^{n(d-1) \eta_{0} P} \min \left\{\frac{|S(\boldsymbol{\alpha} ; P)|}{q^{n P}}, \frac{|S(\boldsymbol{\alpha}+\boldsymbol{\beta} ; P)|}{q^{n P}}\right\}^{2^{d}} \\
& \leqslant C_{0}^{-1} N^{(d-1)}\left(\eta_{0} P ; m^{d} \boldsymbol{\beta}\right) \\
& =C_{0}^{-1} N^{(d-1)}\left(\eta_{0} P ;\left\{m^{d} \boldsymbol{\beta}\right\}\right) \\
& =C_{0}^{-1} N^{\mathrm{aux}}\left(\eta_{0} P ;\left\{m^{d} \boldsymbol{\beta}\right\} t^{d P-\eta_{0} P}\right)
\end{aligned}
$$

by Lemma 5.1. We would now like to apply Corollary 2.6 to estimate the right hand side. Since we are working under the assumption that $\left|\left\{m^{d} \boldsymbol{\beta}\right\} t^{d P-\eta_{0} P}\right| \geqslant q^{d-1}$, we obtain

$$
N^{\text {aux }}\left(\eta_{0} P ;\left\{m^{d} \boldsymbol{\beta}\right\} t^{d P-\eta_{0} P}\right) \leqslant(d-1)^{n} q^{n(d-1) \eta_{0} P} q^{-2^{d+\nu(d)}(d-1) \mathscr{C} \theta_{d}\left(\eta_{0} P\right)}
$$

where $\nu(d)$ is given by (5.2). Note that $\gamma^{-2^{d}}=q^{-\left(N+\varepsilon_{0}\right) 2^{d} \mathscr{C}}=q^{-\eta_{0} P 2^{d} \mathscr{C}-\varepsilon_{0} 2^{d} \mathscr{C}}$. Hence we conclude that

$$
q^{-\varepsilon_{0} 2^{d} \mathscr{C}} \leqslant C_{0}^{-1}(d-1)^{n} q^{\eta_{0} P 2^{d} \mathscr{C}} \cdot q^{-2^{d+\nu(d)}(d-1) \mathscr{C} \theta_{d}\left(\eta_{0} P\right)} .
$$

Suppose first that $d \geqslant 3$, so that $\nu(d)=0$. If $\eta_{0} P \geqslant 2$ then $\theta_{d}\left(\eta_{0} P\right)=\frac{\eta_{0} P}{d-1}+1$ and we may deduce that $q^{-\varepsilon_{0} 2^{d} \mathscr{C}} \leqslant q^{-2^{d}(d-1) \mathscr{C}}$. If $\eta_{0} P=1$ then $\theta_{d}\left(\eta_{0} P\right)=1$ and we may deduce that $q^{-\varepsilon_{0} 2^{d} \mathscr{C}} \leqslant q^{2^{d} \mathscr{C}} \cdot q^{-2^{d}(d-1) \mathscr{C}}=q^{-2^{d}(d-2) \mathscr{C}}$. Neither of these cases is possible, since $\varepsilon_{0}<1 \leqslant d-2$. If $d=2$, on the other hand, then $\nu(d)=1$ and $\theta_{d}\left(\eta_{0} P\right)=\eta_{0} P$, whence $q^{-4 \varepsilon_{0} \mathscr{C}} \leqslant q^{4 \mathscr{C} \eta_{0} P} \cdot q^{-8 \mathscr{C} \eta_{0} P}=q^{-4 \mathscr{C} \eta_{0} P}$. This too is a contradiction.

The case $N=0$. Suppose that alternative (i) holds in Lemma 2.7, with $J=1$ and $\left\{m^{d} \boldsymbol{\beta}\right\}$. Then

$$
q^{-1} \leqslant \max \left\{q^{-d P+d-2}\left\|m^{d} \boldsymbol{\beta}\right\|^{-1},\left\|m^{d} \boldsymbol{\beta}\right\|^{\frac{1}{d-1}} q^{-1}\right\} .
$$

We can't have $q^{-1} \leqslant\left\|m^{d} \boldsymbol{\beta}\right\|^{\frac{1}{d-1}} q^{-1}$, since $\left\|m^{d} \boldsymbol{\beta}\right\|<1$. Thus $q^{-1} \leqslant q^{-d P+d-2}\left\|m^{d} \boldsymbol{\beta}\right\|^{-1}$, whence $1 \leqslant q^{-d P+d-1} \|\left. m^{d} \boldsymbol{\beta}\right|^{-1}$. It follows that

$$
\begin{aligned}
\min \left\{\frac{|S(\boldsymbol{\alpha} ; P)|}{q^{n P}}, \frac{|S(\boldsymbol{\alpha}+\boldsymbol{\beta} ; P)|}{q^{n P}}\right\}^{2^{d}} & =C_{0} q^{-2^{d} \varepsilon_{0} \mathscr{C}} \\
& \leqslant C_{0}\left(q^{-d P+d-1}\left\|m^{d} \boldsymbol{\beta}\right\|^{-1}\right)^{2^{d} \mathscr{C}} \\
& \leqslant C_{0} \max \left\{q^{-d P+d-1}\left\|m^{d} \boldsymbol{\beta}\right\|^{-1},\left\|m^{d} \boldsymbol{\beta}\right\|^{\frac{1}{d-1}} q^{-1}\right\}^{2^{d} \mathscr{C}}
\end{aligned}
$$

This therefore implies Hypothesis 4.1 with $C=C_{0}^{\frac{1}{2 d}}$.
It remains to deal with alternative (ii) of Lemma 2.7, with $J=1$ and $\left\{m^{d} \boldsymbol{\beta}\right\}$. Then $q^{-d P+d} \leqslant\left\|m^{d} \boldsymbol{\beta}\right\| \leqslant q^{-1}$ and $N^{(d-1)}\left(1 ;\left\{m^{d} \boldsymbol{\beta}\right\}\right)=N^{\text {aux }}\left(1 ;\left\{m^{d} \boldsymbol{\beta}\right\} t^{d P-1}\right)$. Taking
$J=1$ in Lemma 5.1, we find that

$$
\begin{aligned}
C_{0} q^{-2^{d} \varepsilon_{0} \mathscr{C}} & =\min \left\{\frac{|S(\boldsymbol{\alpha} ; P)|}{q^{n P}}, \frac{|S(\boldsymbol{\alpha}+\boldsymbol{\beta} ; P)|}{q^{n P}}\right\}^{2^{d}} \\
& \leqslant q^{-n(d-1)} N^{(d-1)}\left(1 ; m^{d} \boldsymbol{\beta}\right) \\
& =q^{-n(d-1)} N^{(d-1)}\left(1 ;\left\{m^{d} \boldsymbol{\beta}\right\}\right) \\
& =q^{-n(d-1)} N^{\mathrm{aux}}\left(1 ;\left\{m^{d} \boldsymbol{\beta}\right\} t^{d P-1}\right) .
\end{aligned}
$$

Since $\nu(d) \geqslant 0$, an application of Corollary 2.6 now yields

$$
\begin{aligned}
C_{0} q^{-2^{d} \varepsilon_{0} \mathscr{C}} & \leqslant q^{-n(d-1)} \cdot(d-1)^{n} q^{n(d-1)} \cdot q^{-2^{d}(d-1) \mathscr{C} \theta_{d}(1)} \\
& =(d-1)^{n} q^{-2^{d}(d-1) \mathscr{C} \theta_{d}(1)} .
\end{aligned}
$$

We have $\theta_{d}(1)=1$ for $d \geqslant 2$ and we see that this is a contradiction, since $C_{0}=(d-1)^{n}$ and $\varepsilon_{0}<1 \leqslant d-1$.
Proof of Theorem 1.3. Let $\mathbb{F}_{q}$ be a fixed finite field of characteristic $>d$. Under the hypotheses of the theorem we have $q \geqslant(d-1)^{n}$. Let $f_{1}, \ldots, f_{R} \in \mathbb{F}_{q}\left[x_{1}, \ldots, x_{n}\right]$ be degree $d$ forms, cutting out a smooth complete intersection in $\mathbb{P}^{n-1}$. In particular $\sigma_{\mathbf{f}} \leqslant R-1$ in Theorem 4.2, by Remark 2.4,

The proof of Theorem 1.3 is deduced from Theorem 4.2 in precisely the same way that Theorem 4.1.2 is deduced from Equation (4.9.1) in [20], the details of which will not be repeated here. We may suppose that we are given $m \in \mathbb{F}_{q}[t] \backslash\{0\}$ and $\mathbf{b} \in \mathbb{F}_{q}[t]^{n}$ with $\operatorname{deg} b_{i}<\operatorname{deg} m$ and $\operatorname{gcd}(\mathbf{b}, m)=1$, and such that the system of equations

$$
f_{1}(m \mathbf{x}+\mathbf{b})=\cdots=f_{R}(m \mathbf{x}+\mathbf{b})=0
$$

is everywhere locally soluble. Under these assumptions it will then suffice to prove that $N(\mathbf{f} ; P, m, \mathbf{b})>0$, for sufficiently large values of $P$, in the notation of (4.1).

We fix the choice of $\mathscr{C}$ in (5.1). Since $q \geqslant(d-1)^{n}$, it follows that

$$
\left(1-\log _{q}(d-1)\right) n \geqslant n-1 .
$$

Hence $\mathscr{C}>d R$ holds under the assumption $n>d(d-1) 2^{d+\nu(d)} R+R$, where $\nu(d)$ is given by (5.2). In the proof of Theorem [1.3, all implied constants are allowed to depend on $q$, on $d, n, R$, and on $m$ and $\mathbf{b}$. In particular, it follows that Hypothesis 4.1 holds with $C=O(1)$. Thus Theorem 4.2 yields $\delta>0$ such that

$$
N(\mathbf{f} ; P, m, \mathbf{b})=\mathfrak{S} \Im q^{(n-d R) P-d R \operatorname{deg} m}+O\left(q^{(n-d R-\delta) P}\right)
$$

Moreover, Lemmas 3.6 and 3.8 yield $\mathfrak{S I}>0$.

## 6. Geometric deductions

We will use the argument deployed in [8] to establish the irreducibility and dimension of the space $\mathrm{M}_{e, b}$ that is the object of Theorem 1.2. This is based on a counting argument over a finite field $k=\mathbb{F}_{q}$ whose characteristic is greater than the degrees $d$
of the forms $f_{1}, \ldots, f_{R} \in \mathbb{F}_{q}\left[x_{1}, \ldots, x_{n}\right]$ that define $X$. We may further assume that $q$ is arbitrarily large. We fix a sufficiently small parameter $\varepsilon>0$ and we henceforth suppose that $q>(d-1)^{1 / \varepsilon}$, so that

$$
\begin{equation*}
0<\frac{\log (d-1)}{\log q}<\varepsilon \tag{6.1}
\end{equation*}
$$

We may further assume that $p_{1}, \ldots, p_{b} \in \mathbb{P}^{1}\left(\mathbb{F}_{q}\right)$ and $y_{1}, \ldots, y_{b} \in X\left(\mathbb{F}_{q}\right)$ in the definition (1.3) of $\mathrm{M}_{e, b}$. After a coordinate change we may also suppose without loss of generality that none of $p_{1}, \ldots, p_{b}$ lie at infinity. In particular, there exist $c_{1}, \ldots, c_{b} \in \mathbb{F}_{q}$ and $\mathbf{a}_{1}, \ldots, \mathbf{a}_{b} \in \mathbb{F}_{q}^{n}$ such that $p_{j}=\left(c_{j}: 1\right)$ and $y_{j}=\left(a_{1, j}: \cdots: a_{n, j}\right)$, for $1 \leqslant j \leqslant b$.
Definition 6.1. Let $N\left(q, e ; \mathbf{a}_{1}, \ldots, \mathbf{a}_{b}\right)$ be the number of tuples $\mathbf{g} \in \mathbb{F}_{q}[t]^{n}$, of degree at most $e$, at least one of degree exactly $e$, with no common zero, such that $f_{1}(\mathbf{g})=$ $\cdots=f_{R}(\mathbf{g})=0$ and $\mathbf{g}\left(c_{j}\right)=\mathbf{a}_{j}$, for $1 \leqslant j \leqslant b$. We simply write $N(q, e)$ for this quantity when $b=0$.
Proposition 6.2. We have

$$
(q-1) \not \mathrm{M}_{e, b}\left(\mathbb{F}_{q}\right)=\sum_{\lambda_{1}, \ldots, \lambda_{b} \in \mathbb{F}_{q}^{*}} N\left(q, e ; \lambda_{1} \mathbf{a}_{1}, \ldots \lambda_{b} \mathbf{a}_{b}\right) .
$$

Proof. Let $\mathrm{N}_{e, b}$ be the set of tuples $\mathbf{g}=\left(g_{1}, \ldots, g_{n}\right) \in \mathbb{F}_{q}[t]^{n}$, of degree at most $e$, at least one of degree exactly $e$, with no common zero, such that $f_{1}(\mathbf{g})=\cdots=f_{R}(\mathbf{g})=0$ and $\mathbf{g}\left(c_{j}\right) \in \mathbb{G}_{m}\left(\mathbb{F}_{q}\right) \mathbf{a}_{j}$, for $1 \leqslant j \leqslant b$. Taking projective coordinates $\left(g_{1}: \cdots: g_{n}\right)$, any $\mathbf{g} \in \mathrm{N}_{e, b}$ defines a degree $e$ map $g: \mathbb{P}^{1} \rightarrow X$ over $\mathbb{F}_{q}$, such that $g\left(p_{j}\right)=y_{j}$ for $1 \leqslant j \leqslant b$. All such maps arise this way, and two tuples define the same map if and only if one is a scalar multiple of the other. Hence

$$
\# \mathrm{~N}_{e, b}=\# \mathbb{G}_{m}\left(\mathbb{F}_{q}\right) \# \mathrm{M}_{e, b}\left(\mathbb{F}_{q}\right)=(q-1) \# \mathrm{M}_{e, b}\left(\mathbb{F}_{q}\right)
$$

Finally, it is clear from the definition that $\# \mathrm{~N}_{e, b}$ is equal to the right hand side of the expression in the statement of the proposition.

Proposition 6.3. We have

$$
\# \mathscr{M}_{0,0}(X, e)\left(\mathbb{F}_{q}\right)=\frac{N(q, e)}{(q-1)\left(q^{3}-q\right)}
$$

Proof. Each point of $\mathscr{M}_{0,0}(X, e)$ corresponds to $\# \mathrm{PGL}_{2}\left(\mathbb{F}_{q}\right)=q^{3}-q$ points of $\operatorname{Mor}_{e}\left(\mathbb{P}^{1}, X\right)$. The result now follows on taking $b=0$ in Proposition 6.2.

We shall prove an explicit upper bound for the counting function defined in Definition 6.1. The following result will be obtained by adapting the proof of Theorem 4.2,

Theorem 6.4. Let $d \geqslant 2$, and $b \geqslant 0$ be integers, and recall the definition (5.2) of $\nu(d)$. Let $f_{1}, \ldots, f_{R} \in \mathbb{F}_{q}\left[x_{1}, \ldots, x_{n}\right]$ be as above, let $c_{1}, \ldots, c_{b} \in \mathbb{F}_{q}$ and let $\mathbf{a}_{1}, \ldots, \mathbf{a}_{b} \in \mathbb{F}_{q}^{n}$. Assume that $n \geqslant\left(d(d-1) 2^{d+\nu(d)+1}+1\right) R$ and

$$
e \geqslant(d+1) b+\max \{1, b R\}
$$

Then

$$
\lim _{q \rightarrow \infty} q^{-(e(n-d R)+(n-R)(1-b))} N\left(q, e ; \mathbf{a}_{1}, \ldots, \mathbf{a}_{b}\right) \leqslant 1
$$

Proof of Theorems 1.1 and 1.2. Assume that $n \geqslant\left(d(d-1) 2^{d+\nu(d)+1}+1\right) R$ and $e \geqslant 1$. Once combined with Proposition 6.3, it follows from [8, Eq. (3.3)] that the case $b=0$ of Theorem 6.4 implies that $\mathscr{M}_{0,0}(X, e)$ is irreducible and of the expected dimension, as required for Theorem 1.1. Appealing to the same method used in [15, p. 2], moreover, it also follows that $\mathscr{M}_{0,0}(X, e)$ is locally a complete intersection, which thereby completes the proof of Theorem 1.1. Arguing along the same lines, we may deduce Theorem 1.2 through the union of Proposition 6.2 and Theorem 6.4.

It remains to prove Theorem [6.4. For now, let $e \geqslant 1$ be an arbitrary integer. In all the estimates that follow, the implied constants are only allowed to depend on $d, n$ and $R$, unless indicated otherwise. We shall drop the coprimality condition from the definition of $N(q, e, b)$, together with the constraint that at least one of the polynomials in the tuple $g_{1}, \ldots, g_{n}$ has exact degree $e$. Moreover, for each index $1 \leqslant j \leqslant b$, the condition $\mathbf{g}\left(c_{j}\right)=\mathbf{a}_{j}$ is equivalent to the congruence condition $\mathbf{g} \equiv \mathbf{a}_{j} \bmod \left(t-c_{j}\right)$. In this way, we see that

$$
N\left(q, e ; \mathbf{a}_{1}, \ldots, \mathbf{a}_{b}\right) \leqslant \widehat{N}(q, e, b)
$$

where

$$
\widehat{N}(q, e, b)=\#\left\{\mathbf{g} \in \mathbb{F}_{q}[t]^{n}: \begin{array}{l}
\operatorname{deg} g_{1}, \ldots, \operatorname{deg} g_{n} \leqslant e \\
f_{1}(\mathbf{g})=\cdots=f_{R}(\mathbf{g})=0 \\
\mathbf{g} \equiv \mathbf{a}_{j} \bmod \left(t-c_{j}\right) \text { for } 1 \leqslant j \leqslant b
\end{array}\right\}
$$

Here we recall that $c_{1}, \ldots, c_{b} \in \mathbb{F}_{q}$ and $\mathbf{a}_{1}, \ldots, \mathbf{a}_{b} \in \mathbb{F}_{q}^{n}$ are given, with $c_{1}, \ldots, c_{b}$ all distinct. Since $\mathbf{a}_{1}, \ldots, \mathbf{a}_{b}$ are vectors representing $\mathbb{F}_{q}$-points on $X$, we also have $f_{k}\left(\mathbf{a}_{j}\right)=0$ for $1 \leqslant k \leqslant R$ and $1 \leqslant j \leqslant b$. Let

$$
m=\left(t-c_{1}\right) \cdots\left(t-c_{b}\right) .
$$

This is a square-free monic polynomial in $\mathbb{F}_{q}[t]$ of degree $b$. It follows from the Chinese remainder theorem that there exists $\mathbf{s} \in \mathbb{F}_{q}[t]^{n}$, with $\operatorname{gcd}(\mathbf{s}, m)=1$ and $\operatorname{deg} s_{i}<b$ for $1 \leqslant i \leqslant n$, such that $f_{i}(\mathbf{s}) \equiv 0 \bmod m$ for $1 \leqslant i \leqslant R$ and

It is now clear that $\hat{N}(q, e, m)=N(\mathbf{f}, e+1-b, m, \mathbf{s})$, in the notation of (4.1). It follows from (4.3) that

$$
\widehat{N}(q, e, m)=\int_{\mathbb{T}^{R}} S(\boldsymbol{\alpha}) \mathrm{d} \boldsymbol{\alpha}
$$

where $S(\boldsymbol{\alpha})=S(\boldsymbol{\alpha} ; e+1-b, m, \mathbf{s})$ is given by (4.2).
Recall the definition of the major arcs of level $J$ that were introduced in (4.6). Taking $P=e+1-b$, these take the shape

$$
\begin{equation*}
\mathfrak{M}(J)=\bigcup_{\substack{g \in \mathbb{F}_{q}[t] \text { monic } \\ 0<|g| \leqslant q^{J}}} \bigcup_{\substack{\mathbf{a} \in \mathbb{F}_{q}[t]^{R} \\|a|<\left|g m^{d}\right| \\ \operatorname{gcd}(\mathbf{a}, g)=1}}\left\{\boldsymbol{\alpha} \in \mathbb{T}^{R}:\left|g m^{d} \boldsymbol{\alpha}-\mathbf{a}\right|<q^{J-d e+d b}\right\} \tag{6.2}
\end{equation*}
$$

Note that $\mathfrak{M}(-1)=\emptyset$. Moreover, the function field version of Dirichlet's theorem in [20, Lemma 4.5.1] shows that for any $\boldsymbol{\alpha} \in \mathbb{T}^{R}$ there exist monic $g \in \mathbb{F}_{q}[t]$ and a vector $\mathbf{a} \in \mathbb{F}_{q}[t]^{R}$, with $|\mathbf{a}|<\left|g m^{d}\right|$ and $\operatorname{gcd}(g, \mathbf{a})=1$, such that $|g| \leqslant q^{J}$ and $\left|g m^{d} \boldsymbol{\alpha}-\mathbf{a}\right|^{R}<q^{-J}$. Hence it follows that $\mathfrak{M}(J)=\mathbb{T}^{R}$ if $J \geqslant R d(e-b) /(R+1)$. We proceed by considering the contribution from the major arc of level $J=0$.

Lemma 6.5. We have

$$
\int_{\mathfrak{M}(0)} S(\boldsymbol{\alpha}) \mathrm{d} \boldsymbol{\alpha}=q^{e(n-d R)+(n-R)(1-b)}\left(1+O\left(q^{-(n-R-1) / 2}\right)\right) .
$$

Proof. We begin by observing that

$$
\begin{aligned}
& \int_{\mathfrak{M}(0)} S(\boldsymbol{\alpha}) \mathrm{d} \boldsymbol{\alpha}=\sum_{\substack{\mathbf{a} \in \mathbb{F}_{q}[t]^{R} \\
|\mathbf{a}|<\left|m^{d}\right|}} \sum_{\substack{\mathbf{g} \in \mathbb{F}_{q}[t]^{n} \\
\mathbf{g} \boldsymbol{g} \mid<q^{++1} \\
\mathbf{g}=\mathrm{s} \bmod m}} \psi\left(\frac{\mathbf{a} \cdot \mathbf{f}(\mathbf{g})}{m^{d}}\right) \int_{|\boldsymbol{\beta}|<q^{-d e}} \psi(\boldsymbol{\beta} . \mathbf{f}(\mathbf{g})) \mathrm{d} \boldsymbol{\beta} \\
& =\sum_{\substack{\mathbf{a} \in \mathbb{F}_{q}[t]^{R} \\
|\mathbf{a}|<\left|m^{d}\right|}} \sum_{\substack{\mathbf{h} \in \mathbb{F}_{q}\left[t n^{n} \\
|\mathbf{h}|<\left|m^{d}\right| \\
\mathbf{h}=\mathrm{s} \bmod m\right.}} \psi\left(\frac{\mathbf{a} \cdot \mathbf{f}(\mathbf{h})}{m^{d}}\right) \sum_{\substack{\mathbf{g} \in \mathbb{F}_{q}\left[t n^{n} \\
|\mathbf{g}|<q^{e+1} \\
\mathbf{g} \equiv \mathbf{h} \bmod m^{d}\right.}} \int_{|\boldsymbol{\beta}|<q^{-d e}} \psi(\boldsymbol{\beta} \cdot \mathbf{f}(\mathbf{g})) \mathrm{d} \boldsymbol{\beta},
\end{aligned}
$$

on breaking into residue classes modulo $m^{d}$. The inner sum over $\mathbf{g}$ becomes

$$
q^{-d R e} \#\left\{\mathbf{g} \in \mathbb{F}_{q}[t]^{n}: \begin{array}{l}
|\mathbf{g}|<q^{e+1}, \mathbf{g} \equiv \mathbf{h} \bmod m^{d} \\
\left|f_{k}(\mathbf{g})\right|<q^{d e} \text { for } 1 \leqslant k \leqslant R
\end{array}\right\}
$$

by [19, Lemma 1(f)]. Any $\mathbf{g}$ in the cardinality to be estimated can be written as $\mathbf{g}=\mathbf{g}_{0} t^{e}+\mathbf{k}$ where $f_{k}\left(\mathbf{g}_{0}\right)=0$, for $1 \leqslant k \leqslant R$, and $\mathbf{k} \in \mathbb{F}_{q}[t]^{n}$ has norm $|\mathbf{k}|<q^{e}$. Recall our assumption that $e \geqslant d b+1$ in the statement of Theorem 6.4. For fixed $\mathbf{g}_{0}$, we must count the number of $\mathbf{k} \in \mathbb{F}_{q}[t]^{n}$ with $|\mathbf{k}|<q^{e}$, such that $\mathbf{k}$ lies in a fixed residue class modulo $m^{d}$. The number of such $\mathbf{k}$ is $\left(q^{e} /\left|m^{d}\right|\right)^{n}=q^{n(e-d b)}$, since $\operatorname{deg} m=b$. Finally, the number of $\mathbf{g}_{0}$ is $q^{n-R}+O\left(q^{(n-R+1) / 2}\right)$ by Deligne's resolution of the Weil conjectures, as presented in Hooley [16, Thm. 2 and Sec. 5], for example. Moreover, the implied constant in this estimate only depends on $d$ and $n$. It therefore follows that

$$
\begin{aligned}
\sum_{\substack{\mathbf{g} \in \mathbb{F}_{q}[t]^{n} \\
|\mathbf{g}| q^{e+1} \\
\mathbf{g} \equiv \mathrm{~h} \bmod m^{d}}} \int_{|\boldsymbol{\beta}|<q^{-d e}} \psi(\boldsymbol{\beta} \cdot \mathbf{f}(\mathbf{g})) \mathrm{d} \boldsymbol{\beta} & =q^{-d R e} \cdot q^{n(e-d b)} \cdot\left(q^{n-R}+O\left(q^{(n-R+1) / 2}\right)\right) \\
& =q^{e(n-d R)+n-d b n-R}\left(1+O\left(q^{-(n-R-1) / 2}\right)\right.
\end{aligned}
$$

Finally, on applying orthogonality of characters, we see that

$$
\sum_{\substack{\mathbf{a} \in \mathbb{F}_{q}[t]^{R} \\|\mathbf{a}|<\left|m^{d}\right|}} \sum_{\substack{\mathbf{h} \in \mathbb{F}_{q}[t]^{n} \\|\mathbf{h}|<\left|m^{d}\right| \\ \mathbf{h} \equiv \mathbf{s} \bmod m}} \psi\left(\frac{\mathbf{a} \mathbf{f}(\mathbf{h})}{m^{d}}\right)=q^{d b R} N^{\dagger}\left(m^{d}\right),
$$

in the notation of (3.11). But it follows from (3.12) that

$$
N^{\dagger}\left(m^{d}\right)=q^{(d-1)(n-R) b} N^{\dagger}(m)=q^{(d-1)(n-R) b}
$$

Combining everything together readily leads to the statement of the lemma.
In order to complete the proof of Theorem [6.4, it therefore suffices to show that

$$
\begin{equation*}
\lim _{q \rightarrow \infty} q^{-(e(n-d R)+(n-R)(1-b))} \sum_{J=0}^{\lceil R d(e-b) /(R+1)\rceil-1}\left|\int_{\mathfrak{M}(J+1) \backslash \mathfrak{M}(J)} S(\boldsymbol{\alpha}) \mathrm{d} \boldsymbol{\alpha}\right|<1 \tag{6.3}
\end{equation*}
$$

Let $\varepsilon>0$ and let $\mathscr{C}$ be given by (5.1), where $\nu(d)$ is given by (5.2). The work in Section 5 shows that Hypothesis 4.1 holds with $C=(d-1)^{\frac{n}{2^{d}}}$. It follows from (6.1) that

$$
\begin{equation*}
\frac{n-R+1}{2^{d+\nu(d)}(d-1)}>\mathscr{C}>\frac{n-R+1}{2^{d+\nu(d)}(d-1)}-\frac{n \varepsilon}{2^{d}(d-1)} . \tag{6.4}
\end{equation*}
$$

Hence, on assuming that $\varepsilon>0$ is chosen to be sufficiently small, it follows that $\mathscr{C}>d R$ if $n \geqslant\left(d(d-1) 2^{d+\nu(d)+1}+1\right) R$, which we henceforth assume. (Although it actually suffices to take $n \geqslant\left(d(d-1) 2^{d+\nu(d)}+1\right) R$ here, this would lead to a significantly worse lower bound for $e$ in Theorem 6.4.)

We need to study the integral

$$
\begin{equation*}
I_{J}=\int_{\mathfrak{M}(J+1) \backslash \mathfrak{M}(J)} S(\boldsymbol{\alpha}) \mathrm{d} \boldsymbol{\alpha} \tag{6.5}
\end{equation*}
$$

where $\mathfrak{M}(J)$ is given by (6.2). We shall proceed differently, according to the size of $J$. Our treatment for larger $J$ is summarised in the following result and is based on the treatment of the minor arcs in Section 4.

Lemma 6.6. Assume that $n \geqslant\left(d(d-1) 2^{d+\nu(d)+1}+1\right) R$, where $\nu(d)$ is given by (5.2). Then

$$
I_{J} \ll q^{\left(n-\frac{n+1-R}{2^{d+\nu(d)}(d-1)}\right)(e+1-b)+O(\varepsilon)}+q^{1+d R b+(n-d R)(e+1-b)-\frac{n-R+1}{2^{d}}\left(1+\left\lfloor\frac{J}{R(d-1)}\right\rfloor\right)+O(\varepsilon)},
$$

where the implied constant in the $O(\varepsilon)$ term depends only on $d, e, n, R$ and $b$.
Proof. The plan is to recycle our treatment of the minor arcs from Section 4 with $P=e+1-b$. We begin by applying Lemma 4.5. Suppose that $\boldsymbol{\alpha} \in \mathbb{T}^{R} \backslash \mathfrak{M}(J)$ and that alternative (ii) holds for some $J^{\prime} \leqslant e+1-b$. Then there exist $g \in \mathbb{F}_{q}[t]$ and $\mathbf{a} \in \mathbb{F}_{q}[t]^{R}$, with $g$ monic and $\operatorname{gcd}(g, \mathbf{a})=1$, such that

$$
0<|g| \leqslant q^{R(d-1)\left(J^{\prime}-1\right)} \quad \text { and } \quad\left|g m^{d} \boldsymbol{\alpha}-\mathbf{a}\right|<q^{-d(e+1-b)+R(d-1) J^{\prime}-(R-1)(d-1)} .
$$

We shall apply this with

$$
\begin{equation*}
J^{\prime}=1+\left\lfloor\frac{J}{R(d-1)}\right\rfloor, \tag{6.6}
\end{equation*}
$$

which would imply that $\boldsymbol{\alpha} \in \mathfrak{M}(J)$, which is a contradiction. We also need to check that $J^{\prime} \leqslant e+1-b$, which certainly holds if

$$
\frac{J}{R(d-1)} \leqslant e-b
$$

Since $J \leqslant\lceil R d(e-b) /(R+1)\rceil-1$, we see that it suffices to have

$$
\frac{R d(e-b)}{R+1} \leqslant(e-b) R(d-1)
$$

which is equivalent to demanding that $d \leqslant(d-1)(R+1)$, an inequality that always holds for $d \geqslant 2$ and $R \geqslant 1$. Hence we find ourselves in alternative (i) of Lemma 4.5 and it follows from Remark 2.4 that

$$
S(\boldsymbol{\alpha}) \ll q^{n(e+1-b)-\frac{(n-R+1) J^{\prime}}{2^{d-1}}}=q^{(n-\delta)(e+1-b)},
$$

for any $\boldsymbol{\alpha} \in \mathfrak{B}=\mathbb{T}^{R} \backslash \mathfrak{M}(J)$, with

$$
\delta=\frac{(n-R+1) J^{\prime}}{(e+1-b) 2^{d-1}}>0
$$

We would now like to apply the second part of Lemma 4.3 with $P=e+1-b$, for which we note that the assumed bound holds with $C_{0} \ll 1$. It remains to check that $\delta(e+1-b) \geqslant \mathscr{C}\left(1-\frac{d b}{d-1}\right)$. If $b \geqslant 1$ then this is trivial, since the right hand side is negative. Assuming that $b=0$, on the other hand, then it follows from (6.4) that it suffices to have

$$
\frac{(n-R+1) J^{\prime}}{2^{d-1}} \geqslant \frac{n-R+1}{2^{d+\nu(d)}(d-1)}
$$

which is valid, since $J^{\prime} \geqslant 1$. We finally deduce from the second part of Lemma 4.3 that

$$
\begin{equation*}
I_{J} \ll q^{(n-\mathscr{C})(e+1-b)}+q^{1+d R b+\left(n-d R-\delta\left(1-\frac{d R}{6}\right)\right)(e+1-b)} . \tag{6.7}
\end{equation*}
$$

Now it follows from (6.4) that

$$
q^{(n-\mathscr{C})(e+1-b)} \leqslant q^{\left(n-\frac{n+1-R}{2^{d+\nu(d)}(d-1)}+O(\varepsilon)\right)(e+1-b)} .
$$

This is satisfactory for the lemma. Next, (6.4) yields

$$
\delta\left(1-\frac{d R}{\mathscr{C}}\right)(e+1-b) \geqslant \frac{(n-R+1) J^{\prime}\left(1-\frac{d R}{\mathscr{C}}\right)}{2^{d-1}}
$$

where

$$
\begin{equation*}
1-\frac{d R}{\mathscr{C}}=1-\frac{d R 2^{d+\nu(d)}(d-1)}{n+1-R}+O(\varepsilon) \geqslant \frac{1}{2}+O(\varepsilon) \tag{6.8}
\end{equation*}
$$

under our assumption on $n$. The lemma follows on recalling the expression (6.6) for $J^{\prime}$ and putting these estimates together in (6.7).

Our second estimate for $I_{J}$ is more appropriate for smaller values of $J$ and draws on the major arc analysis in Section 4 .

Lemma 6.7. Assume that $n \geqslant d R$ and

$$
0 \leqslant J \leqslant e-d b-b-1
$$

Then

$$
\begin{aligned}
& I_{J}<_{\varepsilon} q^{(n-d R)(e+1-b)+d R-R(d-1) b-(J+1)\left(\left(1-\frac{1}{d}\right)\left(\frac{n}{d}-R\right)-1\right)+O(\varepsilon)} \\
& \quad \quad+q^{(n-d R)(e+1-b)-R(d-1) b+\left(1-\frac{2}{d}\right) n+R-(J+d)\left(\left(1-\frac{1}{d}\right)\left(\frac{n}{d}-R\right)-1\right)+O(\varepsilon)},
\end{aligned}
$$

where the implied constant in the $O(\varepsilon)$ term depends only on $d, e, n, R$ and $b$.
Proof. Recall the definition (6.2) of the major arcs. By (4.8), the major arcs of level $J+1$ are non-overlapping if $J+1 \leqslant \frac{1}{2} d(e-b)$. Our assumption $J \leqslant e-d b-b-1$ is sufficient to ensure this property. Hence

$$
\begin{equation*}
I_{J}=\sum_{\substack{g \in \mathbb{F}_{q}[t] \text { monic } \\ 0<|g| \leqslant q^{J+1}}} \sum_{\substack{\mathbf{a} \in \mathbb{F}_{q}[t)^{R} \\|a|<\left|g m^{d}\right| \\ \operatorname{gcd}(\mathbf{a}, g)=1}} \int_{R_{J} \backslash \mathfrak{M}(J)} S(\boldsymbol{\alpha}) \mathrm{d} \boldsymbol{\alpha} \tag{6.9}
\end{equation*}
$$

in (6.5), where

$$
R_{J}=\left\{\boldsymbol{\alpha} \in \mathbb{T}^{R}:\left|\boldsymbol{\alpha}-\frac{\mathbf{a}}{g m^{d}}\right|<\frac{q^{J+1-d e}}{|g|}\right\} .
$$

Since $\boldsymbol{\alpha} \notin \mathfrak{M}(J)$, we may assume in (6.9) that

$$
\begin{equation*}
|g|=q^{J+1} \quad \text { or } \quad|\boldsymbol{\beta}|=\frac{q^{J-d e}}{|g|} \tag{6.10}
\end{equation*}
$$

where $\boldsymbol{\beta}=\boldsymbol{\alpha}-\frac{\mathrm{a}}{g m^{d}}$. We proceed by applying Lemma 4.7, which yields

$$
S(\boldsymbol{\alpha})=q^{n(e+1-b)} S_{g m^{d}}(\mathbf{a}) S_{\infty}\left(m^{d} t^{d(e+1-b)} \boldsymbol{\beta}\right)
$$

provided that the hypotheses of the lemma are met. The first hypothesis is the upper bound $\left|g m^{d}\right| \leqslant q^{e+1-b}$, which is clearly met if $J \leqslant e-d b-b$. The second hypothesis is the bound $|\boldsymbol{\beta}|<q^{-(d-1)(e-b)}|g|^{-1}|m|^{-2 d}$, for which it suffices to have

$$
q^{J+1-d e} \leqslant q^{-(d-1)(e-b)}|m|^{-2 d}=q^{-d e-d b+e-b} .
$$

Thus we see that both hypotheses hold if $J \leqslant e-d b-b-1$, which is what we assumed in the statement of the lemma.

We have therefore shown that

$$
I_{J}=q^{n(e+1-b)} \sum_{\substack{g \in \mathbb{F}_{q}[t] \text { monic } \\ 0<|g| \leqslant q^{J+1}}} \sum_{\substack{\mathbf{a} \in \mathbb{F}_{q}\left[t R^{R} \\|\mathbf{a}|<\left|g m^{d}\right| \\ \operatorname{gcd}(\mathbf{a}, g)=1\right.}} S_{g m^{d}}(\mathbf{a}) \int_{|\boldsymbol{\beta}|<q^{J+1-d e} /|g|} S_{\infty}\left(m^{d} t^{d(e+1-b)} \boldsymbol{\beta}\right) \mathrm{d} \boldsymbol{\beta},
$$

where $g$ and $\boldsymbol{\beta}$ are constrained to satisfy (6.10). Making a change of variables, we see that

$$
\int_{|\boldsymbol{\beta}|<q^{J+1-d e} /|g|} S_{\infty}\left(m^{d} t^{d(e+1-b)} \boldsymbol{\beta}\right) \mathrm{d} \boldsymbol{\beta}=q^{-d R b-d R(e+1-b)} \int_{|\boldsymbol{\gamma}|<q^{J+1+d} /|g|} S_{\infty}(\boldsymbol{\gamma}) \mathrm{d} \boldsymbol{\gamma}
$$

Hence

$$
I_{J}=q^{(n-d R)(e+1-b)-d R b} \sum_{\substack{\left.g \in \mathbb{F}_{q}[t]\right]^{\text {monic }} \\ 0<|g| \leqslant q^{J+1}}} \sum_{\substack{\mathbf{a} \in \mathbb{F}_{[ }[\mid]^{R} \\|\mathbf{a}|<\left|g m^{d}\right| \\ \operatorname{gcd}(\mathbf{a}, g)=1}} S_{g m^{d}}(\mathbf{a}) \int_{|\gamma|<q^{J+1+d} /|g|} S_{\infty}(\boldsymbol{\gamma}) \mathrm{d} \boldsymbol{\gamma},
$$

where $g$ and $\gamma$ are constrained to satisfy

$$
|g|=q^{J+1} \quad \text { or } \quad|\gamma|=\frac{q^{J+d}}{|g|}
$$

Recalling the notation (3.4) for $A(g)$, it follows from the proof of (3.14) that

$$
\sum_{\substack{\mathbf{a} \in \mathbb{F}_{q}[t]^{R} \\\left|\mathbf{a}<\left|g m^{d}\right| \\ \operatorname{gcd}(\mathbf{a}, g)=1\right.}} S_{g m^{d}}(\mathbf{a})=\mathbf{1}_{\operatorname{gcd}(g, m)=1}|m|^{R} A(g) .
$$

Hence, on recalling the notation (3.16) and (3.18), we may now write

$$
\left|I_{J}\right| \leqslant q^{(n-d R)(e+1-b)-R(d-1) b} \sum_{(i, j) \in \mathscr{E}_{1} \cup \mathscr{E}_{2}} \mathscr{S}(i) \cdot|\mathscr{I}(j)|,
$$

where

$$
\begin{aligned}
\mathscr{E}_{1} & =\left\{(J+1, j) \in \mathbb{Z}^{2}: j \leqslant d-1\right\} \\
\mathscr{E}_{2} & =\left\{(i, J+d-i) \in \mathbb{Z}^{2}: 0 \leqslant i \leqslant J+1\right\}
\end{aligned}
$$

Lemma 3.5 implies that

$$
\mathscr{S}(i)<_{\varepsilon} q^{i\left(1-\left(1-\frac{1}{d}\right)\left(\frac{n}{d}-R\right)+O(\varepsilon)\right)},
$$

where the implied constant in $O(\varepsilon)$ term depends only on $d, e, n, R$ and $b$. Likewise, Lemma 3.7 yields

$$
\mathscr{I}(j) \ll \begin{cases}q^{R+j R} & \text { if } j \leqslant d-1, \\ q^{\left(1-\frac{2}{d}\right) n+R-j\left(1-\frac{1}{d}\right)\left(\frac{n}{d}-R\right)} & \text { if } j \geqslant d-1 .\end{cases}
$$

Let $I_{J}^{(k)}$ denote the overall contribution to $I_{J}$ from indices $(i, j) \in \mathscr{E}_{k}$, for $k \in\{1,2\}$. On noting that $j \leqslant d-1$ in $\mathscr{E}_{1}$, we find that

$$
\begin{aligned}
I_{J}^{(1)} & \lll \varepsilon q^{(n-d R)(e+1-b)+R-R(d-1) b} \sum_{(i, j) \in \mathscr{E}_{1}} q^{-i\left(\left(1-\frac{1}{d}\right)\left(\frac{n}{d}-R\right)-1+O(\varepsilon)\right)} \cdot q^{j R} \\
& \lll \varepsilon q^{(n-d R)(e+1-b)+d R-R(d-1) b-(J+1)\left(\left(1-\frac{1}{d}\right)\left(\frac{n}{d}-R\right)-1\right)+O(\varepsilon)} .
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
I_{J}^{(2)} & \lll \varepsilon q^{(n-d R)(e+1-b)-R(d-1) b+\left(1-\frac{2}{d}\right) n+R} \\
& \quad \times \sum_{0 \leqslant i \leqslant J+1} q^{-i\left(\left(1-\frac{1}{d}\right)\left(\frac{n}{d}-R\right)-1+O(\varepsilon)\right)} \cdot q^{-(J+d-i)\left(1-\frac{1}{d}\right)\left(\frac{n}{d}-R\right)} \\
& \ll \varepsilon q^{(n-d R)(e+1-b)-R(d-1) b+\left(1-\frac{2}{d}\right) n+R-(J+d)\left(1-\frac{1}{d}\right)\left(\frac{n}{d}-R\right)+O(\varepsilon)} \sum_{0 \leqslant i \leqslant J+1} q^{i} \\
& \ll \varepsilon q^{(n-d R)(e+1-b)-R(d-1) b+\left(1-\frac{2}{d}\right) n+R-(J+d)\left(\left(1-\frac{1}{d}\right)\left(\frac{n}{d}-R\right)-1\right)+O(\varepsilon)} .
\end{aligned}
$$

Combining these estimates, we therefore complete the proof of the lemma.
It is now time to use this pair of results to prove that (6.3) holds under the assumptions of Theorem 6.4. In particular, we have $d \geqslant 2$ and $n \geqslant\left(d(d-1) 2^{d+\nu(d)+1}+1\right) R$, where $\nu(d)$ is given by (5.2). Moreover, we begin by assuming that $e \geqslant(d+1) b+1$. Write

$$
J_{0}=e-d b-b-1 \quad \text { and } \quad J_{1}=\lceil R d(e-b) /(R+1)\rceil-1,
$$

and note that $0 \leqslant J_{0} \leqslant J_{1} \ll e$.
It follows from Lemma 6.6 that

$$
\begin{aligned}
& \sum_{J=J_{0}+1}^{J_{1}} I_{J}<_{\varepsilon} e q^{\left(n-\frac{n+1-R}{2^{d+\nu(d)}(d-1)}\right)(e+1-b)+O(\varepsilon)} \\
& \quad+e q^{1+d R b+(n-d R)(e+1-b)+O(\varepsilon)} q^{-\frac{n-R+1}{2^{d}}\left(1+\left\lfloor\frac{J_{0}+1}{R(d-1)}\right\rfloor\right)} .
\end{aligned}
$$

Likewise, Lemma 6.7 implies that

$$
\begin{aligned}
\sum_{J=0}^{J_{0}} I_{J}<_{\varepsilon} & q^{(n-d R)(e+1-b)+d R-R(d-1) b+1-\left(1-\frac{1}{d}\right)\left(\frac{n}{d}-R\right)+O(\varepsilon)} \\
& \quad+q^{(n-d R)(e+1-b)-R(d-1) b+\left(1-\frac{2}{d}\right) n+R+d-(d-1)\left(\frac{n}{d}-R\right)+O(\varepsilon)}
\end{aligned}
$$

Putting these together, we may conclude that (6.3) holds, as desired, provided that our parameters $d, n, m, R$ and $e$ are chosen in such a way that the exponents of $q$ are all strictly less than

$$
e(n-d R)+(n-R)(1-b)=(n-d R)(e+1-b)+(d-1) R(1-b)
$$

But, on choosing $\varepsilon>0$ sufficiently small, this is equivalent to demanding the veracity of the four inequalities

$$
\begin{gather*}
\left(n-\frac{n+1-R}{2^{d+\nu(d)}(d-1)}\right)(e+1-b)<(n-d R)(e+1-b)+(d-1) R(1-b),  \tag{6.11}\\
1+d R b-\frac{n-R+1}{2^{d}}\left(1+\left\lfloor\frac{J_{0}+1}{R(d-1)}\right\rfloor\right)<(d-1) R(1-b),  \tag{6.12}\\
1-\left(1-\frac{1}{d}\right)\left(\frac{n}{d}-R\right)<-R, \tag{6.13}
\end{gather*}
$$

and

$$
\begin{equation*}
\left(1-\frac{2}{d}\right) n+R+d-(d-1)\left(\frac{n}{d}-R\right)<(d-1) R . \tag{6.14}
\end{equation*}
$$

Our assumptions on $d$ and $n$ plainly ensure that both (6.13) and (6.14) hold. Moreover, (6.11) holds if and only if

$$
(d-1) R(b-1)<(e+1-b)\left(\frac{n+1-R}{2^{d+\nu(d)}(d-1)}-d R\right) .
$$

Since $e \geqslant(d+1) b+1 \geqslant 2 b+1$ and $n \geqslant\left(d(d-1) 2^{d+\nu(d)+1}+1\right) R$, the right hand side is $>(d-1) R(b-1)$, as required. Thus (6.11) holds.

It remains to check (6.12), for which we note that

$$
1+\left\lfloor\frac{J_{0}+1}{R(d-1)}\right\rfloor \geqslant \frac{J_{0}+2}{R(d-1)}
$$

Recalling that $J_{0}=e-d b-b-1$ and $n \geqslant\left(d(d-1) 2^{d+1}+1\right) R$, we see that (6.12) holds if

$$
2 d(e-d b-b+1) \geqslant 1-R(d-1)+b(2 d-1) R .
$$

Since $d \geqslant 2$, it suffices to have

$$
e \geqslant(d+1+R) b
$$

The proof of Theorem 6.4 is now complete.

## 7. Low degree rational curves

The main goal of this section is to prove Theorem 1.4, Let $X \subset \mathbb{P}^{n-1}$ be a smooth complete intersection over $\mathbb{F}_{q}$, cut out by $R$ hypersurfaces of degree $d$. We shall always assume that $\operatorname{char}\left(\mathbb{F}_{q}\right)>d$ and that $n$ satisfies the bound in (1.1). As usual, we denote by $f_{1}, \ldots, f_{R} \in \mathbb{F}_{q}\left[x_{1}, \ldots, x_{n}\right]$ the relevant forms of degree $d \geqslant 2$.

Recall $N(q, e)$ from Definition 6.1, which counts the number of tuples $\mathbf{g} \in \mathbb{F}_{q}[t]^{n}$, of degree at most $e$, at least one of degree exactly $e$, with no common zero, such that $f_{1}(\mathbf{g})=\cdots=f_{R}(\mathbf{g})=0$. It follows from Proposition 6.3 that

$$
\begin{equation*}
\# \mathscr{M}_{0,0}(X, e)\left(\mathbb{F}_{q}\right)=\frac{N(q, e)}{(q-1)\left(q^{3}-q\right)} . \tag{7.1}
\end{equation*}
$$

In order to prove Theorem 1.4 it will suffice to show that the right hand side is positive if $q$ is sufficiently large in terms of $d$ and $n$, and $e \geqslant 2(d-1) R+3 d$.
7.1. Counting in the affine cone. Let $P \in \mathbb{N}$. For now we simply assume that $q \geqslant(d-1)^{n}$ and we fix the same choice of $\mathscr{C}$ in (5.1). In particular, we have $\mathscr{C}>d R$ under the assumption (1.1), and we saw at the close of Section 5 that Hypothesis 4.1 holds with $C=O(1)$.

We begin by focusing on the counting function $N(\mathbf{f} ; P)=N(\mathbf{f} ; P, 1, \mathbf{0})$ defined in (4.1), with $m=1$ and $\mathbf{b}=\mathbf{0}$. Assume that

$$
\begin{equation*}
P>(d-1) R \tag{7.2}
\end{equation*}
$$

Then it follows from Theorem 4.2 that

$$
\begin{equation*}
N(\mathbf{f} ; P)=\mathfrak{S} \Im q^{(n-d R) P}+O_{\varepsilon}(E(P)), \tag{7.3}
\end{equation*}
$$

for any $\varepsilon>0$, with

$$
E(P)=q^{(n-\mathscr{C}) P}+q^{1+d R+\delta_{0}\left(1-\frac{d R}{6}\right)+\left(n-d R-\delta_{1}\right) P}+q^{\left(n-d R-\delta_{2}+\varepsilon\right) P+n+\delta_{2}(1+R(d-1))}
$$

and where

$$
\begin{equation*}
\delta_{0}=\frac{n-\sigma_{\mathbf{f}}}{(d-1) 2^{d-1} R}, \quad \delta_{1}=\delta_{0}\left(1-\frac{d R}{\mathscr{C}}\right), \quad \delta_{2}=\left(1-\frac{1}{d}\right)\left(\frac{n}{d}-R\right)-1 . \tag{7.4}
\end{equation*}
$$

Here, the implicit constant depends only on $d, n, R$ and $\varepsilon$.
The singular series $\mathfrak{S}$ is defined in (3.3) and the singular integral $\mathfrak{I}$ in (3.21). We shall need to develop precise estimates for these quantities.

Lemma 7.1. Assume $n>2(d-1) R$ and that $q$ is sufficiently large with respect to $d$ and $n$. Then $\mathfrak{S}=1+O\left(q^{-\frac{n-R-3}{2}}\right)$.

Proof. For any prime $\pi \in \mathbb{F}_{q}[t]$, define

$$
\sigma(\pi)=\sum_{j=0}^{\infty} \sum_{\substack{|\mathbf{a}|<\left|\pi^{j}\right| \\ \operatorname{gcd}(\mathbf{a}, \pi)=1}} S_{\pi^{j}}(\mathbf{a}),
$$

in the notation of (3.2). In the light of the notation (3.4), and the assumed lower bound $n>2(d-1) R$, it follows from Lemma 3.2 that this sum converges. Moreover, we clearly have

$$
\mathfrak{S}=\prod_{\pi} \sigma(\pi)
$$

in (3.3). Using Ramanujan sums and recalling that $m=1$, it is routine to check that

$$
\sigma(\pi)=\lim _{k \rightarrow \infty}|\pi|^{-k(n-R)} N\left(\pi^{k}\right)
$$

in the notation of (3.7), where

$$
N\left(\pi^{k}\right)=\#\left\{\mathbf{z} \in \mathbb{F}_{q}[t]^{n}:|\mathbf{z}|<\left|\pi^{k}\right|, f_{i}(\mathbf{z}) \equiv 0 \bmod \pi^{k} \text { for } 1 \leqslant i \leqslant R\right\}
$$

It will also be convenient to recall that

$$
N^{*}\left(\pi^{k}\right)=\#\left\{\mathbf{z} \in \mathbb{F}_{q}[t]^{n}:|\mathbf{z}|<\left|\pi^{k}\right|, \pi \nmid \mathbf{z}, f_{i}(\mathbf{z}) \equiv 0 \bmod \pi^{k} \text { for } 1 \leqslant i \leqslant R\right\}
$$

We proceed by adapting the proof of Lemma 3.2 to prove that

$$
\begin{equation*}
N\left(\pi^{e}\right)=\frac{N^{*}(\pi)}{|\pi|^{n-R}} C_{\infty}(\pi)|\pi|^{(n-R) e}+O\left(|\pi|^{\left(1-\frac{1}{d}\right) e n}\right) \tag{7.5}
\end{equation*}
$$

where

$$
C_{\infty}(\pi)=\sum_{j \geqslant 0}|\pi|^{-j(n-d R)} .
$$

First, we deduce from (3.8) that $N^{*}\left(\pi^{k}\right)=|\pi|^{(n-R)(k-1)} N^{*}(\pi)$, for any $k \geqslant 1$. It now follows from (3.10) that

$$
\begin{aligned}
N\left(\pi^{e}\right) & =N^{*}(\pi)|\pi|^{(n-R)(e-1)} \sum_{0 \leqslant j<e / d}|\pi|^{(d-1) j n-(n-R) d j}+O\left(|\pi|^{\left(1-\frac{1}{d}\right) e n}\right) \\
& =N^{*}(\pi)|\pi|^{(n-R)(e-1)} \sum_{0 \leqslant j<e / d}|\pi|^{-j(n-d R)}+O\left(|\pi|^{\left(1-\frac{1}{d}\right) e n}\right),
\end{aligned}
$$

for $e>d$. It is clear that

$$
\left.\left.\left|C_{\infty}(\pi)-\sum_{0 \leqslant j<e / d}\right| \pi\right|^{-j(n-d R)}|\ll| \pi\right|^{-\frac{e}{d}(n-d R)} .
$$

Putting these together therefore establishes the claimed bound (7.5).
Note that

$$
\begin{equation*}
C_{\infty}(\pi)=1+O\left(|\pi|^{-(n-d R)}\right) \tag{7.6}
\end{equation*}
$$

Moreover, according to (3.9), we have $N^{*}(\pi)=|\pi|^{n-R}\left(1+O\left(|\pi|^{-\frac{n-R-1}{2}}\right)\right)$. It now follows from (7.5) and (7.6) that

$$
\begin{aligned}
\sigma(\pi) & =\lim _{k \rightarrow \infty}|\pi|^{-k(n-R)}\left(\frac{N^{*}(\pi)}{|\pi|^{n-R}} C_{\infty}(\pi)|\pi|^{(n-R) k}+O\left(|\pi|^{\left(1-\frac{1}{d}\right) k n}\right)\right) \\
& =\frac{N^{*}(\pi)}{|\pi|^{n-R}} C_{\infty}(\pi) \\
& =1+J(\pi)
\end{aligned}
$$

where $J(\pi)=|\pi|^{-\frac{n-R-1}{2}}$. Moreover, the implicit constant depends only on $d$ and $n$. Note that

$$
\sum_{\pi} J(\pi) \ll \sum_{\pi} q^{-\left(\frac{n-R-1}{2}\right) \operatorname{deg} \pi} \ll \sum_{j=1}^{\infty} q^{j-\left(\frac{n-R-1}{2}\right) j} \ll q^{-\frac{n-R-3}{2}}
$$

Let us assume $q$ is sufficiently large with respect to $d$ and $n$, so that we can use Taylor expansion to deduce that $\log (1+J(\pi))=O(|J(\pi)|)$, for an absolute implied constant. Then it follows that

$$
\begin{aligned}
\mathfrak{S} & =\prod_{\pi}(1+J(\pi)) \\
& =\exp \left(\sum_{\pi} \log (1+J(\pi))\right) \\
& =\exp \left(O\left(\sum_{\pi}|J(\pi)|\right)\right) \\
& =\exp \left(O\left(q^{-\frac{n-R-3}{2}}\right)\right) \\
& =1+O\left(q^{-\frac{n-R-3}{2}}\right)
\end{aligned}
$$

where we again used the fact that $q$ is sufficiently large to make use of the Taylor series expansion of exp.
Lemma 7.2. Assume $n \geqslant 2(d-1) R$. Then $\mathfrak{I}=q^{R(d-1)}\left(1+O\left(q^{-\frac{n-R-1}{2}}\right)\right)$.
Proof. Let $T>2 d-1$. Firstly, it follows from Lemma 3.8 that

$$
|\mathfrak{I}-\mathfrak{I}(T)| \ll q^{n-\left(1-\frac{1}{d}\right)\left(\frac{n}{d}-R\right) T} .
$$

It also follows from combining the proof of Lemma 3.7 with (7.5) that

$$
\begin{aligned}
\mathfrak{I}(T) & =q^{R T} m(T) \\
& =q^{(d-1-T) n+R T} N\left(t^{T+1-d}\right) \\
& =q^{(d-1-T) n+R T}\left(\frac{N^{*}(t)}{q^{n-R}} C_{\infty}(t) q^{(n-R)(T+1-d)}+O\left(q^{\left(1-\frac{1}{d}\right)(T+1-d) n}\right)\right) \\
& =\frac{N^{*}(t)}{q^{n-R}} C_{\infty}(t) q^{R(d-1)}+O\left(q^{-\frac{T}{d}(n-d R)-\frac{1}{d}(1-d) n}\right) .
\end{aligned}
$$

But (3.9) and (7.6) yield

$$
\frac{N^{*}(t)}{q^{n-R}} C_{\infty}(t)=1+O\left(q^{-\frac{n-R-1}{2}}\right)
$$

under the assumption on $n$ in the lemma. Therefore, on choosing $T$ to be sufficiently large, we easily arrive at the desired estimate.
7.2. Möbius inversion. Now that we have an estimate for $N(\mathbf{f} ; P)$ in (7.3), we may go on to deduce an estimate for the quantity

$$
N^{*}(\mathbf{f} ; P)=\#\left\{\mathbf{g} \in \mathbb{F}_{q}[t]_{\text {prim }}^{n}: 1 \leqslant|\mathbf{g}|<q^{P}, f_{1}(\mathbf{g})=\cdots=f_{R}(\mathbf{g})=0\right\}
$$

in which $\mathbf{g} \in \mathbb{F}_{q}[t]_{\text {prim }}^{n}$ means that $\operatorname{gcd}\left(g_{1}, \ldots, g_{n}\right)=1$. As usual we shall assume that $q$ is sufficiently large in terms of $d$ and $n$. We shall also need to assume that $P$ satisfies the inequality (7.2).

Let $\mu: \mathbb{F}_{q}[t] \rightarrow\{0, \pm 1\}$ denote the function field version of the Möbius function. Then Möbius inversion yields

$$
\begin{aligned}
N^{*}(\mathbf{f} ; P) & =\sum_{\substack{k \in \mathbb{F}_{q}[t] \\
k \text { monic } \\
|k|<q^{P}}} \mu(k) \#\left\{\mathbf{g} \in \mathbb{F}_{q}[t]^{n}: 1 \leqslant|\mathbf{g}|<q^{P-\operatorname{deg} k}, f_{1}(\mathbf{g})=\cdots=f_{R}(\mathbf{g})=0\right\} \\
& =\sum_{\substack{k \in \mathbb{F}_{q}[t] \\
k \text { monic } \\
|k|<q^{P}}} \mu(k)(N(\mathbf{f} ; P-\operatorname{deg} k)-1) .
\end{aligned}
$$

We will need to truncate the sum over $k$, so that $P-\operatorname{deg} k$ satisfies the constraint (7.2). Thus

$$
N^{*}(\mathbf{f} ; P)=\sum_{\substack{k \in \mathbb{F}_{q}[t] \\ k \text { monic } \\|k|<q^{P-(d-1) R}}} \mu(k) N(\mathbf{f} ; P-\operatorname{deg} k)+O\left(q^{P}+L(P)\right)
$$

where

$$
L(P)=\sum_{\substack{k \in \mathbb{F}_{q}[t] \\ k \text { moni } \\|k| \geqslant q^{P-(d i c-1) R}}} N(\mathbf{f} ; P-\operatorname{deg} k) \leqslant \sum_{\substack{k \in \mathbb{F}_{q}[t] \\ k \text { monic } \\|k| \geqslant q^{P-(d-1) R}}}\left(\frac{q^{P}}{|k|}\right)^{n} \ll q^{(d-1) R(n-1)+P} .
$$

On appealing to (7.3), we therefore deduce that

$$
N^{*}(\mathbf{f} ; P)=\mathfrak{S} \Im q^{(n-d R) P} \sum_{\substack{k \in \mathbb{F}_{q}[t] \\ k \text { monic } \\|k|<q^{P-(d-1) R}}} \mu(k) q^{-(n-d R) \operatorname{deg} k}+O\left(q^{(d-1) R(n-1)+P}+E(P)\right),
$$

since $\sum_{k} E(P-\operatorname{deg} k) \ll E(P)$.
We see that

$$
\sum_{\substack{k \in \mathbb{F}_{q}[t] \\ k \\ k \text { monic } \\|k|<q^{P-(d-1) R}}} \mu(k) q^{-(n-d R) \operatorname{deg} k}=1+O\left(\sum_{1 \leqslant j<P} q^{j-(n-d R) j}\right)=1+O\left(q^{1-(n-d R)}\right)
$$

Our assumption (1.1) implies that $n>2(d-1) R$. Hence, on combining these estimates with Lemmas 7.1 and 7.2 , we obtain

$$
\begin{aligned}
N^{*}(\mathbf{f} ; P) & =\mathfrak{S} \mathfrak{I} q^{(n-d R) P}\left(1+O\left(q^{1-(n-d R)}\right)\right)+O\left(q^{(d-1) R(n-1)+P}+E(P)\right) \\
& =q^{(n-d R) P+(d-1) R}\left(1+O\left(q^{-\frac{n-R-3}{2}}\right)\right)+O\left(q^{(d-1) R(n-1)+P}+E(P)\right) \\
& =q^{(n-d R) P+(d-1) R}(1+O(\widetilde{E}(P)))
\end{aligned}
$$

where

$$
\begin{aligned}
\widetilde{E}(P)= & q^{(d-1) R(n-2)-(n-d R-1) P}+q^{-\frac{n-R-3}{2}}+q^{(d R-\mathscr{C}) P-(d-1) R} \\
& +q^{1+R+\delta_{0}\left(1-\frac{d R}{\mathscr{G}}\right)-\delta_{1} P}+q^{-\delta_{2}(P-1-(d-1) R)+\varepsilon P+n-(d-1) R} .
\end{aligned}
$$

7.3. Completion of the proof. We are now ready to conclude the proof of Theorem 1.4, based on the expression (7.1). Assuming that $q$ is sufficiently large with respect to $d$ and $n$, it is clear from our work above that

$$
\begin{aligned}
N(q, e) & =N^{*}(\mathbf{f} ; e+1)-N^{*}(\mathbf{f} ; e) \\
& =q^{(n-d R)(e+1)+(d-1) R}\left(1-q^{-(n-d R)}\right)+O\left(q^{(n-d R)(e+1)+(d-1) R} \widetilde{E}(e+1)\right),
\end{aligned}
$$

under the assumption that $e>(d-1) R$.
Clearly $n-d R>0, \mathscr{C}>d R$ and $\frac{n-R-3}{2}>0$ under our assumption on $n$ in (1.1). Thus, in the light of our remarks on the size of $q$, we may conclude that $\# \mathscr{M}_{0,0}(X, e)\left(\mathbb{F}_{q}\right)>0$ if the three inequalities

$$
\begin{align*}
(n-d R-1)(e+1) & >(d-1) R(n-2), \\
\delta_{1}(e+1) & >1+R+\delta_{0}\left(1-\frac{d R}{\mathscr{C}}\right),  \tag{7.7}\\
\delta_{2}(e-(d-1) R)+(d-1) R & >\varepsilon(e+1)+n,
\end{align*}
$$

all hold, for any $\varepsilon>0$. Recalling (7.4), the second inequality is equivalent to

$$
e>\frac{1+R}{\delta_{1}}
$$

Now it follows from our assumption (1.1) that $n \geqslant d(d-1) 2^{d+\nu(d)+1} R+R$, where $\nu(d)$ is given by (5.2). In particular $1-\frac{d R}{\mathscr{G}} \geqslant \frac{1}{2}+O(\varepsilon)$, for any $\varepsilon>0$, by (6.8). But then, since Remark 2.4yields $\sigma_{\mathrm{f}} \leqslant R-1$, it follows from (7.4) that $\delta_{1} \geqslant d 2^{\nu(d)+1}+O(\varepsilon) \geqslant d$, on choosing $\varepsilon$ sufficiently small. We conclude that the second inequality in (7.7) is satisfied if $e \geqslant R+1$. This is implied by our assumption $e>(d-1) R$. We now make the more stringent assumption $e \geqslant 2(d-1) R$, under which the first inequality is obvious. Finally, we make the even more stringent assumption that $e \geqslant 2(d-1) R+3 d$ and we check the third inequality in (7.7) with $\varepsilon=\delta_{2} \varepsilon^{\prime}$, for any $\varepsilon^{\prime}>0$, which is then equivalent to

$$
\delta_{2}\left(e\left(1-\varepsilon^{\prime}\right)-(d-1) R-\varepsilon^{\prime}\right)+(d-1) R>n
$$

It follows from (7.4) that

$$
\delta_{2}=\left(1-\frac{1}{d}\right)\left(\frac{n}{d}-R\right)-1 \geqslant \frac{n-d R-2 d}{2 d}
$$

since $1-\frac{1}{d} \geqslant \frac{1}{2}$ for $d \geqslant 2$. Hence, since $e \geqslant 2(d-1) R+3 d$, we obtain

$$
\delta_{2}\left(e\left(1-\varepsilon^{\prime}\right)-(d-1) R-\varepsilon^{\prime}\right) \geqslant \frac{n-d R-2 d}{2 d} \cdot\left(\left(1-2 \varepsilon^{\prime}\right)(d-1) R+\left(1-\varepsilon^{\prime}\right) 3 d-\varepsilon^{\prime}\right) .
$$

Taking $R \geqslant 1$ and $\varepsilon^{\prime}$ sufficiently small, it therefore follows that

$$
\delta_{2}\left(e\left(1-\varepsilon^{\prime}\right)-(d-1) R-\varepsilon^{\prime}\right)+(d-1) R \geqslant \frac{(4 d-1)(n-d R-2 d)}{2 d}-O\left(\varepsilon^{\prime}\right) .
$$

This is greater than $n$ under our assumptions on $n$. This therefore completes the proof of Theorem 1.4.

## References

[1] R. Beheshti and N. M. Kumar, Spaces of rational curves on complete intersections. Compositio Math. 149 (2013), 1041-1060. 2
[2] B.J. Birch, Forms in many variables. Proc. Roy. Soc. Ser. A 265 (1961/62), 245-263. 3, 711
[3] T.D. Browning, R. Dietmann and D.R. Heath-Brown, Rational points on intersections of cubic and quadric hypersurfaces. J. Inst. Math. Jussieu 14 (2015), 703-749. 7
[4] T.D. Browning and D.R. Heath-Brown, Rational points on quartic hypersurfaces. J. reine angew. Math. 629 (2009), 37-88. 12
[5] T.D. Browning and D.R. Heath-Brown, Forms in many variables and differing degrees. J. Eur. Math. Soc. 9 (2017), 357-394. 7
[6] T.D. Browning and W. Sawin, A geometric version of the circle method. Ann. of Math. 191 (2020), 893-948. 28
[7] T.D. Browning and W. Sawin, Free rational curves on low degree hypersurfaces and the circle method. Algebra $\mathcal{E}^{2}$ Number Theory 17 (2023), 719-748. 2, 3
[8] T.D. Browning and P. Vishe, Rational curves on smooth hypersurfaces of low degree. Algebra § Number Theory 11 (2017), 1657-1675. 2, 3, 39,41
[9] A. Cafure and G. Matera, Improved explicit estimates on the number of solutions of equations over a finite field. Finite Fields Appl. 12 (2006), 155-185. 4
[10] I. Coskun and J. Starr, Rational curves on smooth cubic hypersurfaces. Int. Math. Res. Not. 24 (2009), 4626-4641. 2
[11] O. Debarre, Higher-dimensional algebraic geometry. Springer-Verlag, 2001. 2
[12] J. Glas, Rational points on complete intersections of cubic and quadric hypersurfaces over $\mathbb{F}_{q}(t)$. Preprint, 2023. (arXiv:2306.02718) 7
[13] M. Greenberg, Lectures on forms in many variables. Benjamin, New York, 1969. 3
[14] W. Fulton, Intersection theory. 2nd ed., Springer, 1998. 10
[15] J. Harris, M. Roth and J. Starr, Rational curves on hypersurfaces of low degree. J. reine angew. Math. 571 (2004), 73-106. 141
[16] C. Hooley, On the number of points on a complete intersection over a finite field. J. Number Theory 38 (1991), 338-358. 16, 42
[17] B. Kim and R. Pandharipande, The connectedness of the moduli space of maps to homogeneous spaces. Symplectic geometry and mirror symmetry, Chapter 5, World Scientific, 2001. 2
[18] J. Kollár, Looking for rational curves on cubic hypersurfaces. Higher-dimensional geometry over finite fields (NATO Science for Peace Security Series D: Information and Communication Security 16), 92-122, editors D. Kaledin and Y. Tschinkel, IOS, Amsterdam, 2008. 4
[19] R. M. Kubota, Waring's Problem for $\mathbb{F}_{q}[x]$. Dissertationes Math. (Rozprawy Mat.) 117 (1974), 60 pp . 7,42
[20] S.A. Lee, On the applications of the circle method to function fields, and related topics. Ph.D. thesis, University of Bristol, 2013. 3, 7, 15, 21, 23, 28, 31, 36, 39, 42
[21] A. Mânzățeanu, Rational curves on cubic hypersurfaces over finite fields. Mathematika 67 (2021), 366-387. 3, 20
[22] D. Mumford, The red book of varieties and schemes. 2nd ed., SLNM 1358, Springer, 1999. 11
[23] E. Riedl and D. Yang, Kontsevich spaces of rational curves on Fano hypersurfaces. J. reine angew. Math. 748 (2019), 207-225. 2
[24] S. Rydin Myerson, Quadratic forms and systems of forms in many variables. Invent. Math. 213 (2018), 205-235. 3, 8, 23, 24, 25
[25] S. Rydin Myerson, Systems of cubic forms in many variables. J. reine angew. Math. 757 (2019), 309-328. 3

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