8.3 Lagrange form for the remainder

There is a more convenient expression for the remainder term in Taylor's theorem. The **Lagrange form for the remainder** is

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}, \qquad \text{for some } \; c \in (a,x).$$

To prove this expression for the remainder we will first need to prove the following lemma:

Lemma
Let $h(t)$ be differentiable $n+1$ times on $[a,x]$ with $h^{(k)}(a) = 0$ for $0 \leq k \leq n$ and $h(x) = 0$. Then $\exists \; c \in (a,x)$ s.t. $h^{(n+1)}(c) = 0$.

Proof:
$h(a) = 0 = h(x)$ so by Rolle's theorem $\exists \; c_1 \in (a,x)$ s.t. $h'(c_1) = 0$.
$h'(a) = 0 = h'(c_1)$ so by Rolle's theorem $\exists \; c_2 \in (a,c_1)$ s.t. $h''(c_1) = 0$.
Repeating this argument a total of n times we arrive at
$h^{(n)}(a) = 0 = h^{(n)}(c_n)$ so by Rolle's theorem $\exists \; c_{n+1} \in (a,c_n)$ s.t. $h^{(n+1)}(c_{n+1}) = 0$.
This proves the lemma with $c = c_{n+1} \in (a,x)$.

Proof of the Lagrange form of the remainder:
Consider the function

$$h(t) = (f(t) - P_n(t))(x-a)^{n+1} - (f(x) - P_n(x))(t-a)^{n+1}.$$

By construction $h(x) = 0$.
Also $\frac{d^k}{dt^k}(t-a)^{n+1}$ is zero when evaluated at $t = a$ for $0 \leq k \leq n$. Furthermore, by definition of the Taylor polynomial $P_n(t)$ we have that $f^{(k)}(a) = P_n^{(k)}(a)$ for $0 \leq k \leq n$.
Hence $h^{(n)}(a) = 0$ for $0 \leq k \leq n$.
$h(t)$ therefore satisfies the conditions of the lemma and we have that
$\exists \; c \in (a,x)$ s.t. $h^{(n+1)}(c) = 0$.
As $P_n(t)$ is a polynomial of degree n then $P_n^{(n+1)}(t) = 0$.
Also, $\frac{d^{n+1}}{dt^{n+1}}(t-a)^{n+1} = (n+1)!$ hence

$$0 = h^{(n+1)}(c) = (x-a)^{n+1} f^{(n+1)}(c) - (n+1)!(f(x) - P_n(x))$$

Rearranging this expression gives the required result

$$f(x) = P_n(x) + \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}.$$

One use of the Lagrange form of the remainder is to provide an upper bound on the error of a Taylor polynomial approximation to a function.
Suppose that $|f^{(n+1)}(t)| \leq M, \quad \forall\ t$ in the closed interval between a and x.
Then $|R_n(x)| \leq \frac{M|x-a|^{n+1}}{(n+1)!}$ provides a bound on the error.

Eg. Show that the error in approximating e^x by its 6^{th} order Taylor polynomial is always less than 0.0006 throughout the interval $[0,1]$.

In this example $f(x) = e^x$ and $n = 6$ so we first require an upper bound on $|f^{(7)}(t)| = |e^t|$ for $t \in [0,1]$. As e^t is monotonic increasing and positive then $|e^t| \leq e^1 = e < 3$. Thus, in the above notation, we may take $M = 3$.
As $a = 0$ we now have that $|R_6(x)| < \frac{3|x|^7}{7!} \leq \frac{3}{7!}$ for $x \in [0,1]$.
Evaluating $\frac{3}{7!} = \frac{1}{1680} < 0.0006$ and the required result has been shown.