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Riemannian Geometry IV

Solutions, set 5.

Exercise 10.

(a) Let $p = (X, Y, Z) \in \mathbb{W}^2$. Then

$$L_p = \{(X, Y, Z) + t(X, Y, Z + 1) \mid t \in \mathbb{R}\}$$

and

$$L_p \cap \mathbb{B}^2 = (X, Y, Z) - \frac{Z}{Z+1}(X, Y, Z + 1) = (\frac{X}{Z+1}, \frac{Y}{Z+1}, 0).$$

We conclude that

$$f(X, Y, Z) = \left(\frac{X}{Z+1}, \frac{Y}{Z+1}, 0\right) \in \mathbb{B}^2.$$

Conversely, let $f^{-1}(x, y, 0) = (X, Y, Z)$ with $X^2+Y^2 = Z^2-1$. Since $x = \frac{X}{Z+1}$, $y = \frac{Y}{Z+1}$, we conclude that

$$x^2 + y^2 = \frac{X^2 + Y^2}{(Z + 1)^2} = \frac{Z^2 - 1}{(Z + 1)^2} = \frac{Z - 1}{Z + 1},$$

so $Z = \frac{1+x^2+y^2}{1-x^2-y^2}$ and $Z + 1 = \frac{2}{1-x^2-y^2}$. We conclude that

$$\begin{aligned} f^{-1}(x, y, 0) = (X, Y, Z) &= ((Z + 1)x, (Z + 1)y, Z) \\ &= \left(\frac{2x}{1 - x^2 - y^2}, \frac{2y}{1 - x^2 - y^2}, \frac{1 + x^2 + y^2}{1 - x^2 - y^2}\right). \end{aligned}$$

(b) We have

$$\psi^{-1}(y_1, y_2) = f \circ \varphi^{-1}(y_1, y_2) = \left(\frac{\sinh y_2}{1 + \cosh y_2} \cos y_1, \frac{\sinh y_2}{1 + \cosh y_2} \sin y_1, 0\right).$$

(c) Let $p = \varphi^{-1}(x_1, x_2) = (\cos x_1 \sinh x_2, \sin x_1 \sinh x_2, \cosh x_2)$. Then $f(p) = \psi^{-1}(x_1, x_2)$, so $y_1 = x_1$, $y_2 = x_2$. So

$$f(p) = \left(\frac{\sinh x_2}{1 + \cosh x_2} \cos x_1, \frac{\sinh x_2}{1 + \cosh x_2} \sin x_1, 0 \right).$$

Moreover, we have

$$\begin{aligned} \frac{\partial}{\partial y_1}\Big|_{f(p)} &= \frac{1}{1 + \cosh x_2} \left(-\sinh x_2 \sin x_1, \sinh x_2 \cos x_1, 0 \right), \\ \frac{\partial}{\partial y_2}\Big|_{f(p)} &= \frac{1}{1 + \cosh x_2} \left(\cos x_1, \sin x_1, 0 \right). \end{aligned}$$

We know from Example 14 (a) that

$$\begin{aligned} \left\langle \frac{\partial}{\partial x_1}\Big|_p, \frac{\partial}{\partial x_1}\Big|_p \right\rangle_p &= \sinh^2 x_2, \\ \left\langle \frac{\partial}{\partial x_1}\Big|_p, \frac{\partial}{\partial x_2}\Big|_p \right\rangle_p &= 0, \\ \left\langle \frac{\partial}{\partial x_2}\Big|_p, \frac{\partial}{\partial x_2}\Big|_p \right\rangle_p &= 1. \end{aligned}$$

We obviously have

$$\left\langle \frac{\partial}{\partial y_1}\Big|_{f(p)}, \frac{\partial}{\partial y_2}\Big|_{f(p)} \right\rangle_{f(p)} = 0.$$

Since

$$\|f(p)\|_0^2 = \frac{\sinh^2 x_2}{(1 + \cosh x_2)^2},$$

(where $\| \cdot \|_0$ denotes the standard Euclidean norm,) we conclude that

$$1 - \|f(p)\|_0^2 = \frac{2}{1 + \cosh x_2}.$$

We calculate

$$\begin{aligned} &\left\langle \frac{\partial}{\partial y_1}\Big|_{f(p)}, \frac{\partial}{\partial y_1}\Big|_{f(p)} \right\rangle_{f(p)} = \\ &\quad \frac{4}{(1 - \|f(p)\|_0^2)^2} \frac{1}{(1 + \cosh x_2)^2} \|(-\sinh x_2 \sin x_1, \sinh x_2 \cos x_1, 0)\|_0^2 = \\ &\qquad \sinh^2 x_2 \sin^2 x_1 + \sinh^2 x_2 \cos^2 x_1 = \sinh^2 x_2. \end{aligned}$$

Similarly, we obtain

$$\left\langle \frac{\partial}{\partial y_2}\Big|_{f(p)}, \frac{\partial}{\partial y_2}\Big|_{f(p)} \right\rangle_{f(p)} =$$
$$\frac{4}{(1-\|f(p)\|_0^2)^2} \frac{1}{(1+\cosh x_2)^2} \|(\cos x_1, \sin x_1, 0)\|_0^2 =$$
$$\cos^2 x_1 + \sin^2 x_1 = 1.$$

Exercise 11.

(a) We calculate

$$Df_A(z)(v) = \frac{d}{dt}\Big|_{t=0} f_A(z+tv) = \frac{d}{dt}\Big|_{t=0} \frac{a(z+tv)+b}{c(z+tv)+d}$$
$$= \frac{av(cz+d) = (az+b)cv}{(cz+d)^2} = \frac{(ad-bc)v}{(cz+d)^2} = \frac{1}{(cz+d)^2}v,$$

where we used in the last step that $\det A = ad - bc = 1$. The imaginary part of $f_A(z)$ is related to the imaginary part of z in the following way:

$$\mathrm{Im}(f_A(z)) = \mathrm{Im}\left(\frac{az+b}{cz+d}\right) = \mathrm{Im}\left(\frac{(az+b)(c\bar{z}+d)}{|cz+d|^2}\right)$$
$$= \frac{1}{|cz+d|^2}\mathrm{Im}(adz + bc\bar{z} + ac|z|^2 + bd) = \frac{(ad-bc)\mathrm{Im}(z)}{|cz+d|} = \frac{\mathrm{Im}(z)}{|cz+d|^2}.$$

We conclude that

$$g_{f_A(z)}\left(Df_A(z)(v), Df_A(z)(v)\right) = \frac{1}{\mathrm{Im}(f_A(z))^2}\frac{1}{|cz+d|^4}\langle v, v\rangle$$
$$= \frac{|cz+d|^4}{\mathrm{Im}(z)}\frac{1}{|cz+d|^4}\langle v, v\rangle = \frac{1}{\mathrm{Im}(z)}\langle v, v\rangle = g_z(v, v),$$

so f_A is an isometry.

(b) One easily checks that $f_{A_1}(z) = z + x$ and $f_{A_2}(z) = yz$ for $A_1 = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ and $A_2 = \begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix}$. The group action property tells us then that

$$f_{A_1A_2}(i) = f_{A_1} \circ f_{A_2}(i) = f_{A_1}(iy) = x + iy \in \mathbb{H}^2,$$

which means that $\{f_A(i) \mid A \in SL(2,\mathbb{R})\} = \mathbb{H}^2$.

(c) First let $A = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}$ be an arbitrary matrix in $SO(2)$. Then

$$f_A(i) = \frac{i\cos\alpha + \sin\alpha}{-i\sin\alpha + \cos\alpha} = (\sin\alpha + i\cos\alpha)(\cos\alpha + i\sin\alpha) = i.$$

Conversely, let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be such that

$$f_A(i) = \frac{ai+b}{ci+d} = i.$$

Then $ai+b = -c+di$, i.e., $a = d$ and $b = -c$. Moreover, $1 = ad-bc = a^2+b^2$. Therefore, there exists $\alpha \in [0,2\pi)$ such that $a = \cos\alpha$ and $b = \sin\alpha$ and

$$A = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \in SO(2).$$

Exercise 12. T^2 is the rotation of a circle in the (X,Z)-plane with centre $(R,0,0)$ and radius $r > 0$ around the vertical Z-axis.

We first note that

$$\begin{aligned} \frac{\partial}{\partial x_1}\Big|_{\varphi^{-1}(x_1,x_2)} &= (-r\sin x_1\cos x_2, -r\sin x_1\sin x_2, r\cos x_1), \\ \frac{\partial}{\partial x_2}\Big|_{\varphi^{-1}(x_1,x_2)} &= (-(R+r\cos x_1)\sin x_2, (R+r\cos x_1)\cos x_2, 0). \end{aligned}$$

This implies that

$$\big(g_{ij}(\varphi^{-1}(x_1,x_2))\big) = \begin{pmatrix} r^2 & 0 \\ 0 & (R+r\cos x_1)^2 \end{pmatrix}.$$

So $\sqrt{\det(g_{ij}(\varphi^{-1}(x_1,x)2)))} = r(R+r\cos x_1)$. Since $\varphi^{-1}(V) = U \subset T^2$ covers all of T^2 expect a set of measure zero, we obtain

$$\begin{aligned} \mathrm{vol}(T^2) = \mathrm{vol}(U) = \int_V r(R+r\cos x_1)dx_1dx_2 \\ = \int_0^{2\pi}\int_0^{2\pi} r(R+r\cos x_1)dx_1dx_2 = 4\pi^2 rR. \end{aligned}$$

Exercise 13. The formula for the hyperbolic volume yields

$$\begin{aligned}
\mathrm{vol}(\Delta P_1P_2P_3) &= \int_{x_1}^{x_2}\int_{\sqrt{r^2-(x-x_0)^2}}^{\infty}\frac{dy}{y^2}dx = \int_{x_1}^{x_2}\frac{1}{y}\Big|_{\infty}^{\sqrt{r^2-(x-x_0)^2}}dx \\
&= \int_{x_1}^{x_2}\frac{1}{\sqrt{r^2-(x-x_0)^2}}dx = \frac{1}{r}\int_{x_1}^{x_2}\frac{1}{\sqrt{1-\left(\frac{x-x_0}{r}\right)^2}}dx \\
&= \frac{1}{r}\int_{x_1-x_0}^{x_2-x_0}\frac{1}{\sqrt{1-\left(\frac{x}{r}\right)^2}}dx = \int_{\frac{x_1-x_0}{r}}^{\frac{x_2-x_0}{r}}\frac{dy}{\sqrt{1-y^2}} \\
&= \arcsin\left(\frac{x_2-x_0}{r}\right)-\arcsin\left(\frac{x_1-x_0}{2}\right) = \arcsin\left(\frac{x_2-x_0}{r}\right)+\arcsin\left(\frac{x_0-x_1}{2}\right).
\end{aligned}$$

The picture implies that $\cos\alpha = \frac{x_0-x_1}{r}$. Since $\alpha \in (0,\pi/2)$, we have

$$\alpha = \arccos\left(\frac{x_0-x_1}{r}\right).$$

Similarly, we conclude that

$$\beta = \arccos\left(\frac{x_2-x_0}{r}\right).$$

Using $\arcsin(x)+\arccos(x)=\pi/2$ we finally obtain the desired formula:

$$\begin{aligned}
\mathrm{vol}(\Delta P_1P_2P_3) &= \arcsin\left(\frac{x_2-x_0}{r}\right)+\arcsin\left(\frac{x_0-x_1}{2}\right) \\
&= \pi-\arccos\left(\frac{x_2-x_0}{r}\right)-\arccos\left(\frac{x_0-x_1}{2}\right) = \pi-\alpha-\beta.
\end{aligned}$$