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Riemannian Geometry IV

Solutions, set 14.

Exercise 32. (a) Let $x_n \in M$ be a Cauchy sequence. We have to show that x_n is convergent in M. By compactness of (M, d_g), we know that there exists a convergent subsequence x_{n_j} of x_n. Let $x := \lim x_{n_j} \in M$. It remains to show that $x_n \to x$. Let $\epsilon > 0$ be given. Since x_n is Cauchy, there exists a $N > 0$ such that we have for all $n, m \geq N$: $d(x_n, x_m) < \epsilon/2$. Since $x_{n_j} \to x$, there exists an index $n_J \geq N$ such that $d(x_{n_J}, x) < \epsilon/2$. Both results imply for $n \geq N$:

$$d(x_n, x) \leq d(x_n, x_{n_J}) + d(x_{n_J}, x) < \epsilon/2 + \epsilon/2 = \epsilon.$$

This shows that $x_n \to x$.

(b) Since (M, d_g) is complete, it is geodesically complete, by Hopf-Rinow. Therefore, for every start vector $v \in TM$, there exists a geodesic $c : \mathbb{R} \to M$ with domain equal to all of $\mathbb{R}$ with $c'(0) = v$. This implies that Φ_t is defined on all of TM, since $\Phi_t(v) = c'(t)$.

Exercise 33. (a) Note that $[fX, Y] = f[X, Y] - (Yf)X$. We have

$$\begin{aligned} R(fX, Y)Z &= \nabla_{fX}\nabla_Y Z - \nabla_Y \nabla_{fX} Z - \nabla_{[fX,Y]} Z = \\ &= f\nabla_X \nabla_Y Z - \nabla_Y(f\nabla_X Z) - \nabla_{f[X,Y]-(Yf)X} Z = \\ &= f\nabla_X\nabla_Y Z - (Yf)\nabla_X Z - f\nabla_Y\nabla_X Z - f\nabla_{[X,Y]} Z + (Yf)\nabla_X Z = \\ &= f(\nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]} Z) = fR(X, Y)Z. \end{aligned}$$

(b) Using the symmetry $R(X, Y)Z = -R(Y, X)Z$, we conclude with (a) that

$$R(X, fY)Z = -R(fY, X)Z = -fR(Y, X)Z = fR(X, Y)Z.$$

(c) Using the symmetry $\langle R(X, Y)Z, W\rangle = \langle R(Z, W)X, Y\rangle$ twice, we conclude with (a) that

$$\begin{aligned} \langle R(X, Y)fZ, W\rangle &= \langle R(fZ, W)X, Y\rangle = \langle fR(Z, W)X, Y\rangle = \\ &= f\langle R(Z, W)X, Y\rangle = f\langle R(X, Y)Z, W\rangle = \langle fR(X, Y)Z, W\rangle. \end{aligned}$$

(d) Since (c) holds for all vector fields W, we conclude that

$$R(X,Y)fZ = fR(X,Y)Z.$$

Using this, together with (a) and (b), we obtain

$$R(fX,gY)hZ = fghR(X,Y)Z.$$

Exercise 34. We have

$$\begin{aligned}
R(X,Y)Z + R(Y,Z)X + R(Z,X)Y &= (\nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z)+\\
&+ (\nabla_Y\nabla_Z X - \nabla_Z\nabla_Y X - \nabla_{[Y,Z]}X) + \nabla_Z\nabla_X Y - \nabla_X\nabla_Z Y - \nabla_{[Z,X]}Y) =\\
&= \nabla_X(\nabla_Y Z - \nabla_Z Y) + \nabla_Y(\nabla_Z X - \nabla_X Z) + \nabla_Z(\nabla_X Y - \nabla_Y X) -\\
&- (\nabla_{[X,Y]}Z) + \nabla_{[Y,Z]}X) + \nabla_{[Z,X]}Y) =\\
&= \nabla_X[Y,Z] + \nabla_Y[Z,X] + \nabla_Z[X,Y] - (\nabla_{[X,Y]}Z) + \nabla_{[Y,Z]}X) + \nabla_{[Z,X]}Y) =\\
&= (\nabla_X[Y,Z] - \nabla_{[Y,Z]}X) + (\nabla_Y[Z,X] - \nabla_{[Z,X]}Y) + (\nabla_Z[X,Y] - \nabla_{[X,Y]}Z) =\\
&= -([[Y,Z],X] + [[Z,X],Y] + [[X,Y],Z]) = 0.
\end{aligned}$$

Exercise 35. Induction proof for

$$\left(\nabla_{\frac{\partial}{\partial x_i}} F_k\right)(p) = 0, \tag{1}$$

$$\nabla_{\frac{\partial}{\partial x_i}} \langle F_k, F_k\rangle^{-1/2}(p) = 0, \tag{2}$$

$$\left(\nabla_{\frac{\partial}{\partial x_i}} E_k\right)(p) = 0, \tag{3}$$

for all $i \in \{1,\ldots,n\}$.

One easily checks (1), (2), (3) for $k = 1$. Assume all three equations hold for k. Then we obtain

$$\left(\nabla_{\frac{\partial}{\partial x_i}} F_{k+1}\right)(p) = \left(\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_{k+1}}\right)(p) - \frac{\partial}{\partial x_i}\Big|_p \sum_{j=1}^{k} \left\langle \frac{\partial}{\partial x_{k+1}}, E_j \right\rangle E_j.$$

Using at the right hand side the product rule, the Riemannian property of the Levi-Civita connection, and the induction hypothesis $\nabla_{\frac{\partial}{\partial x_i}} E_j(p) = 0$ for $1 \le j \le k$, we conclude that the whole expression vanishes. Next, we obtain

$$\nabla_{\frac{\partial}{\partial x_i}} \langle F_{k+1}, F_{k+1}\rangle^{-1/2}(p) = -\frac{1}{\|F_{k+1}(p)\|^3}\langle \nabla_{\frac{\partial}{\partial x_i}} F_{k+1}, F_{k+1}\rangle(p),$$

which implies that also this expression vanishes because of (1). Finally,

$$\left(\nabla_{\frac{\partial}{\partial x_i}} E_{k+1}\right)(p) = \nabla_{\frac{\partial}{\partial x_i}} \langle F_{k+1}, F_{k+1}\rangle^{-1/2}(p) F_{k+1}(p) + \frac{1}{\|F_{k+1}(p)\|}\left(\nabla_{\frac{\partial}{\partial x_i}} F_{k+1}\right)(p),$$

which vanishes again because of (1) and (2). This finishes the induction procedure.

We conclude

$$\left(\nabla_{E_i} E_j\right)(p) = \nabla_{E_i(p)} E_j = 0$$

from (3), since $E_i(p)$ is just a linear combination of the basis vectors $\frac{\partial}{\partial x_k}$.