Lecture 2

In the last lecture we introduced mathematical statements and connectives. We explained how to find out whether a combined mathematical statement is true or false using truth tables. In this lecture we will continue with a bit of logic and also introduce **basic facts about sets***...*

Let us first introduce another connective of statements: "if and only if", often abbreviated by "iff". The mathematical symbol for this is $\Leftrightarrow$. The statement $A \Leftrightarrow B$ is only true if both statements A and B have the same truth values. The truth table looks as follows:

A	B	A $\Leftrightarrow$ B
False	False	True
False	True	False
True	False	False
True	True	True

Therefore, if we say that two statements A and B are equivalent, we can also say that the statement $A \Leftrightarrow B$ is true. Let us now rewrite some fundamental logic laws with this symbol.

Theorem. *Let A, B, C be arbitrary statements. Then the following statements are true.*

(a) Laws of Commutativity:

$$\begin{aligned} A \text{ and } B &\quad\Leftrightarrow\quad B \text{ and } A, \\ A \text{ or } B &\quad\Leftrightarrow\quad B \text{ or } A. \end{aligned}$$

(b) Laws of Associativity:

$$\begin{aligned} (A \text{ and } B) \text{ and } C &\quad\Leftrightarrow\quad A \text{ and } (B \text{ and } C), \\ (A \text{ or } B) \text{ or } C &\quad\Leftrightarrow\quad A \text{ or } (B \text{ or } C). \end{aligned}$$

(c) Laws of Distributivity:

$$\begin{aligned} (A \text{ and } B) \text{ or } C &\quad\Leftrightarrow\quad (A \text{ or } C) \text{ and } (B \text{ or } C), \\ (A \text{ or } B) \text{ and } C &\quad\Leftrightarrow\quad (A \text{ and } C) \text{ or } (B \text{ and } C). \end{aligned}$$

(d) De Morgan's Rules:

$$\begin{aligned} (\text{ not } A) \text{ and } (\text{ not } B) &\quad\Leftrightarrow\quad \text{not } (A \text{ or } B), \\ (\text{ not } A) \text{ or } (\text{ not } B) &\quad\Leftrightarrow\quad \text{not } (A \text{ and } B). \end{aligned}$$

Proof. We only prove the second equivalence in (d) with the help of a truth table:

A	B	not A	not B	(not A) or (not B)	A and B	not (A and B)
False	False	True	True	True	False	True
False	True	True	False	True	False	True
True	False	False	True	True	False	True
True	True	False	False	False	True	False

□

Next, we look at another important concept: **Sets** as **collections of objects**. Here are some frequently used symbols in connection with sets:

- Sets are usually described using $\{\cdots\}$ brackets. Here is a list of the most frequently used sets of numbers and vectors:

  natural numbers $\mathbb{N} := \{1, 2, 3, 4, \dots\}$,

  integer numbers $\mathbb{Z} := \{\dots, -2, -1, 0, 1, 2, \dots\}$,

  rational numbers $\mathbb{Q} := \{\frac{p}{q} \mid p \in \mathbb{Z} \text{ and } q \in \mathbb{N}\}$,

  real numbers $\mathbb{R}$,

  complex numbers $\mathbb{C} = \{x + iy \mid x, y \in \mathbb{R}\}$,

  n-dimensional real vectors $\mathbb{R}^n = \{(x_1, \dots, x_n) \mid x_1, \dots, x_n \in \mathbb{R}\}$.

  You will learn in Linear Algebra that the set $\mathbb{R}^n$ carries the additional structure of a vector space.

- x is an *element* of the set X: $x \in X$.

- x is *not an element* of the set X: $x \notin X$.

- Y is a *subset* of X: $Y \subset X$.

- As already used above, a set X is often described in the form

  $$X = \{x \mid x \text{ has certain properties}\}.$$

  Here is an example:

  $$\{n \in \mathbb{N} \mid n \text{ is an odd square}\} = \{1, 9, 25, 49, \dots\}.$$

- *Union* of two sets X and Y: $X \cup Y = \{x \mid x \in X \text{ and } x \in Y\}$.

- *Intersection* of two sets X and Y: $X \cap Y = \{x \mid x \in X \text{ or } x \in Y\}$.

- *Difference* of two sets X and Y: $X\backslash Y = \{x \mid x \in X \text{ and } x \notin Y\}$.
- *Complement* of a set X within a bigger set Z: $X^c = Z\backslash X$.
- We have $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$ and, e.g.,

$$\sqrt{2} \in \mathbb{R}\backslash\mathbb{Q}, \quad i \in \mathbb{C}\backslash\mathbb{R}.$$

- The *empty set* $\emptyset = \{\}$. Note the empty set is a subset of every set.
- Two sets X, Y are called *disjoint* if $X \cap Y = \emptyset$. Here is an example:

$$\mathbb{R} \cap \mathbb{R}^2 = \emptyset.$$

- Let $a < b$ be two real numbers. Then we have the following intervals:

$$\begin{aligned} (a,b) &:= \{x \in \mathbb{R} \mid a < x < b\}, \\ (a,b] &:= \{x \in \mathbb{R} \mid a < x \leq b\}, \\ [a,b) &:= \{x \in \mathbb{R} \mid a \leq x < b\}, \\ [a,b] &:= \{x \in \mathbb{R} \mid a \leq x \leq b\}. \end{aligned}$$

  We also have unbounded intervals, for example,

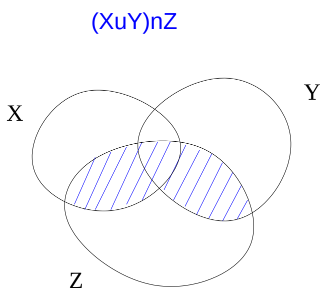
$$(-\infty, b] := \{x \in \mathbb{R} \mid x \leq b\}.$$

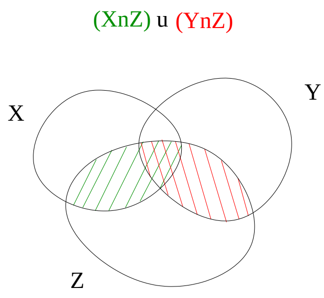
  Usually, when $-\infty$ or ∞ appears as bound of an interval, we use the round brackets "(" and ")" there, since $-\infty$ and ∞ are not proper real numbers and, therefore, should normally **not be included** into the interval.

Set operations can be illustrated by **Venn Diagrams**.

Example: Illustration of the distributive law

$$(X \cup Y) \cap Z = (X \cap Z) \cup (Y \cap Z).$$





But be aware: **Venn Diagrams cannot replace rigorous proofs** (see Question 5 in the Set Problems of Week 2).

Strategies to prove a set inclusion and to prove the equality of two sets:

How to prove $X \subset Y$? You need to show that every element of X is also an element of Y.

How to prove $X = Y$? This is often done by showing $X \subset Y$ and $Y \subset X$.

Examples:

(a) Show that $X = Y$, where

$$\begin{aligned} X &:= \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}, \\ Y &:= \{(\cos t, \sin t) \mid t \in \mathbb{R}\}. \end{aligned}$$

The inclusion $Y \subset X$ is easy to show: Let $(x,y) \in Y$, i.e., $(x,y) = (\cos t, \sin t)$ for some $t \in \mathbb{R}$. Then

$$x^2 + y^2 = \cos^2 t + \sin^2 t = 1,$$

i.e., $(x,y) \in X$.

Conversely: Let $(x,y) \in X$, i.e., $x^2 + y^2 = 1$. Then $-1 \leq x \leq 1$ and, since cos maps $\mathbb{R}$ onto $[-1,1]$, there exists $t \in \mathbb{R}$ such that $x = \cos t$, and we have

$$y^2 = 1 - x^2 = 1 - \cos^2 t = \sin^2 t.$$

This implies that $y = \pm \sin t$. In the case $y = \sin t$, we have $(x,y) = (\cos t, \sin t) \in Y$ and we are done. If $y = -\sin t$, we choose $s = -t \in \mathbb{R}$ and have

$$(\cos s, \sin s) = (\cos(-t), \sin(-t)) = (\cos t, -\sin t) = (x,y).$$

This shows that $(x,y) \in Y$.

(b) Show that $X \subset Y_1 \cup Y_2$, where

$$\begin{aligned} X &:= \{n^2 \mid n \in \mathbb{Z}\}, \\ Y_1 &:= \{4k \mid k \in \mathbb{Z}\}, \\ Y_2 &:= \{4k+1 \mid k \in \mathbb{Z}\}. \end{aligned}$$

Let $n \in \mathbb{Z}$. Then n can be even or odd, i.e., $n = 2l$ or $n = 2l + 1$ for some $l \in \mathbb{Z}$. In the first case

$$n^2 = (2l)^2 = 4l^2 = 4k$$

with $k = l^2 \in \mathbb{Z}$, i.e., $n^2 \in Y_1$. In the second case

$$n^2 = (2l + 1)^2 = 4l^2 + 4l + 1 = 4(l^2 + l) + 1 = 4k + 1$$

with $k = l^2 + l \in \mathbb{Z}$, i.e., $n^2 \in Y_2$.