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Geometry III/IV

Problems, set 3.

Exercise 7. Find a fundamental domain for the symmetry group of the pattern of Exercise 4.

Exercise 8. The purpose of this exercise is to show that every finite subgroup of $O(2)$ is either isomorphic to the finite cyclic group $C_k = \langle r \mid r^k = e \rangle$ of order k or to the the dihedral group $D_k = \langle r, s \mid r^k = s^2 = e, srs = r^{-1} \rangle$ of order $2k$. The following steps might be helpful:

(a) Let $G \subset SO(2)$ be a non-trivial finite group and $r := R_\alpha \in G$ with minimal $\alpha \in (0, 2\pi)$. Show that there is an integer $k \geq 2$ such that $\alpha = \frac{2\pi}{k}$ and that G is generated by r. Consequently, G is isomorphic to C_k.

(b) Let $G \subset O(2)$ with at least one element $s \in G$ not in $SO(2)$. Let $G_0 = G \cap SO(2)$ and $G_1 = G \backslash G_0$. Show that the map $f : G_0 \to G$, $f(g) = sg$ maps G_0 bijectively to G_1, so that we have $|G| = 2|G_0|$. From (a) we know that G_0 is generated by a rotation r of order $k \geq 1$ (by the angle $\alpha = 2\pi/k$), so G is generated by r, s and the elements of G are given as

$$G = \{e, r, \dots, r^{k-1}, s, sr, \dots, sr^{k-1}\}.$$

Moreover, s must be a reflection along a line passing through the origin.

(c) Show that, in the case of (b), $srs = r^{-1}$. Then G satisfies the same relations as the dihedral group D_k and has the same number of elements. Therefore, both groups must be isomorphic.

Note that the webpage for Geometry III/IV is at

`http://www.maths.dur.ac.uk/~dma0np/geom0809/geometry.html`