Differential Geometry III (Math 3021)

Solutions to the Second Homework 13.12.2012

Question 17 Let $L_{(u,v)} \subset \mathbb{R}^3$ be the straight line through $(u,v,0)$ and $(0,0,1)$, i.e.,

$$L_{(u,v)} = \{(0,0,1) + t(u,v,-1) \mid t \in \mathbb{R}\}.$$

We like to find the intersection points $(tu, tv, 1-t) \in L_{(u,v)} \cap S^2(1)$. This means that

$$(tu)^2 + (tv)^2 + (1-t)^2 = 1,$$

i.e., $t(t(1+u^2+v^2)-2) = 0$. The choice $t = 0$ leads to the intersection point $(0,0,1) + 0(u,v,-1) = (0,0,1)$, and the choice $t = 2/(1+u^2+v^2)$ to the intersection point

$$(0,0,1) + \frac{2}{1+u^2+v^2}(u,v,-1) = \left(\frac{2u}{1+u^2+v^2}, \frac{2v}{1+u^2+v^2}, \frac{u^2+v^2-1}{1+u^2+v^2}\right).$$

This shows that we have

$$x(u,v) = \left(\frac{2u}{1+u^2+v^2}, \frac{2v}{1+u^2+v^2}, \frac{u^2+v^2-1}{1+u^2+v^2}\right).$$

The map is obviously continuous. By the geometric construction, we see that x is a bijective map from $\mathbb{R}^2$ to $S^2(1)\backslash\{(0,0,1)\}$. In order to show that it is a homeomorphism, we calculate the inverse map $S^2(1)\backslash\{(0,0,1)\} \to \mathbb{R}^2$. Let

$$\left(S^2(1)\backslash\{(0,0,1)\}\right) \ni (X,Y,Z) = \left(\frac{2u}{1+u^2+v^2}, \frac{2v}{1+u^2+v^2}, \frac{u^2+v^2-1}{1+u^2+v^2}\right).$$

We want to express (u,v) in terms of (X,Y,Z). We see that $1 - Z = 2/(1+u^2+v^2)$ and, consequently, $u = X/(1-Z)$ and $v = Y/(1-Z)$. On $S^2(1)\backslash\{(0,0,1)\}$, we obviously have $Z \neq 1$, and $(u,v) = (X/(1-Z), Y/(1-Z))$ is well defined. Moreover, this map is obviously continuous (as composition of continuous functions).

Finally, we need to show that x_u, x_v are linearly independent. We have

$$\begin{aligned} x_u &= \frac{2}{(1+u^2+v^2)^2}(1-u^2+v^2, -2uv, 2u), \\ x_v &= \frac{2}{(1+u^2+v^2)^2}(-2uv, 1+u^2-v^2, 2v). \end{aligned}$$

Now, observe that

$$\begin{vmatrix} 1-u^2+v^2 & -2uv \\ 2u & 2v \end{vmatrix} = 2v(1+u^2+v^2).$$

Similarly, the determinant of two other corresponding entries of x_u and x_v leads to $2u(1+u^2+v^2)$. So if $(u,v) \neq 0$, at least one of these two determinants is nonzero, i.e., the vectors x_u, x_v are linearly independent. In the case $(u,v) = (0,0)$, we have

$$x_u(0,0) = (2,0,0), \qquad x_v(0,0) = (0,2,0),$$

which are again two linearly independent vectors.

Question 19 (i) We have $\operatorname{grad} f(x,y,z) = 2(x+y+z-1)(1,1,1)$. Therefore, $\operatorname{grad} f(x,y,z) = 0$ if and only if $x+y+z=1$, in which case we have $f(x,y,z) = (x+y+z-1)^2 = 0$.

(ii) We have $f(\mathbb{R}^3) = [0,\infty)$. Since every value $c \in (0,\infty)$ is a regular value, the set $\{x \in \mathbb{R}^3 \mid f(x) = c\}$ is a surface. In the case $c = 0$, we have

$$\{x \in \mathbb{R}^3 \mid f(x) = 0\} = \{(x,y,z) \in \mathbb{R}^3 \mid x+y+z = 1\},$$

and this is obviously a plane through $(1/3, 1/3, 1/3)$, orthonormal to $(1,1,1)$, i.e., again a surface.

(iii) For $c > 0$, the set determined by $f(x) = c$ is the union of two planes, orthogonal to $(1,1,1)$ and satisfying

$$E_\pm = \{x+y+z = 1 \pm \sqrt{c}\},$$

i.e., E_+ contains the point $\frac{1}{3}(1+\sqrt{c})(1,1,1)$ and E_- contains the point $\frac{1}{3}(1-\sqrt{c})(1,1,1)$.

Question 29 (i) We have

$$\begin{aligned} x_u(u,v) &= (-\cosh(v)\sin(u), \cosh(v)\cos(u), 0), \\ x_v(u,v) &= (\sinh(v)\cos(u), \sinh(v)\sin(u), 1). \end{aligned}$$

This implies that

$$\begin{aligned} E(u,v) &= (-\cosh)^2(v)\sin^2(u) + \cosh^2(v)\cos^2(u) = \cosh^2(v), \\ F(u,v) &= 0, \\ G(u,v) &= \sinh^2(v)\cos^2(u) + \sinh^2(v)\sin^2(u) + 1 = \sinh^2(v) + 1 = \cosh^2(v), \end{aligned}$$

i.e., the first fundamental form at $x(u,v)$ is just a multiple of the standard inner product in $\mathbb{R}^2$ by the factor $\cosh^2(v)$.

(ii) We have

$$\begin{aligned} \widetilde{x}_u(u,v) &= (-\sinh(v)\cos(u), -\sinh(v)\sin(u), -1), \\ \widetilde{x}_v(u,v) &= (-\cosh(v)\sin(u), \cosh(v)\cos(u), 0). \end{aligned}$$

This implies that

$$\begin{aligned} \widetilde{E}(u,v) &= (-\sinh)^2(v)\cos^2(u) + (-\sinh)^2(v)\sin^2(u) + (-1)^2 = \cosh^2(v), \\ \widetilde{F}(u,v) &= 0, \\ \widetilde{G}(u,v) &= (-\cosh)^2(v)\sin^2(u) + \cosh^2(v)\cos^2(u) = \cosh^2(v), \end{aligned}$$

i.e., the first fundamental form at $\widetilde{x}(u,v)$ is again just a multiple of the standard inner product in $\mathbb{R}^2$ by the factor $\cosh^2(v)$.

(iii) Now we choose

$$y_\theta(u,v) = \cos(\theta)x(u,v) + \sin(\theta)\widetilde{x}(u,v).$$

We obviously have

$$\begin{aligned} (y_\theta)_u &= \cos(\theta)x_u + \sin(\theta)\widetilde{x}_u, \\ (y_\theta)_v &= \cos(\theta)x_v + \sin(\theta)\widetilde{x}_v. \end{aligned}$$

We easily check that $\langle x_u, \widetilde{x}_u\rangle = 0 = \langle x_v, \widetilde{x}_v\rangle$ and

$$\langle x_u, \widetilde{x}_v\rangle + \langle x_v, \widetilde{x}_u\rangle = \cosh^2(v) - (\sinh^2(v) + 1) = 0.$$

This implies that

$$\begin{aligned} \langle (y_\theta)_u, (y_\theta)_u\rangle &= \cos^2(\theta)E + \sin^2(\theta)\widetilde{E} + 2\sin(\theta)\cos(\theta)\langle x_u, \widetilde{x}_u\rangle = \cosh^2(v), \\ \langle (y_\theta)_u, (y_\theta)_v\rangle &= \cos^2(\theta)F + \sin^2(\theta)\widetilde{F} + \sin(\theta)\cos(\theta)(\langle x_u, \widetilde{x}_v\rangle + \langle x_v, \widetilde{x}_u\rangle) \\ &= \cos^2(\theta)\cdot 0 + \sin^2(\theta)\cdot 0 + \sin(\theta)\cos(\theta)\cdot 0 = 0, \\ \langle (y_\theta)_v, (y_\theta)_v\rangle &= \cos^2(\theta)G + \sin^2(\theta)\widetilde{G} + 2\sin(\theta)\cos(\theta)\langle x_v, \widetilde{x}_v\rangle = \cosh^2(v), \end{aligned}$$

i.e., the first fundamental form at $y_\theta(u,v)$ is again just a multiple of the standard inner product in $\mathbb{R}^2$ by the factor $\cosh^2(v)$.

Question 36 The computation of the length of α is based on the explicit formulas $\alpha(t) = (c\cos t, c\sin t)$ and $\alpha'(t) = (-c\sin t, c\cos t)$. Therefore

$$\|\alpha'(t)\|^2_{\alpha(t)} = \frac{c^2}{c^2\sin^2 t} = \frac{1}{\sin^2 t}.$$

This implies that

$$L(c) = \int_{\pi/6}^{5\pi/6} \|\alpha'(t)\|_{\alpha(t)}\, dt = \int_{\pi/6}^{5\pi/6} \frac{1}{\sin t}\, dt.$$

To calculate the intersection point of $\alpha(t) = (\sqrt{2}\cos t, \sqrt{2}\sin t)$ with the curve $\beta(s) = (1,s)$, we have to find $s > 0$ with $1 + s^2 = (\sqrt{2})^2$. This leads to $s = 1$, and shows that $\beta(1) = (1,1) = \alpha(\pi/4)$. The angle ϕ of intersection between α and β at $(1,1)$ satisfies

$$\cos\phi = \frac{\langle \alpha'(\pi/4), \beta'(1)\rangle_{(1,1)}}{\|\alpha'(\pi/4)\|_{(1,1)}\cdot\|\beta'(1)\|_{(1,1)}} = \frac{\langle(-\sqrt{2},\sqrt{2}),(0,1)\rangle}{\|(-\sqrt{2},\sqrt{2})\|\cdot\|(0,1)\|} = \frac{1}{\sqrt{2}}.$$

Since $0 \le \phi \le \pi$, we conclude that $\phi = \pi/4$.