Analysis III/IV (Math 3011, Math 4201)

Solutions to Exercise Sheet 7 23.11.2011

1. Let $f(x,y) = x^2y^3$. Then we have

$$df = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy = 2xy^3\,dx + 3x^2y^2\,dy = \omega.$$

Using $\int_c df = f(c(b)) - f(c(a))$ for piecewise smooth curves $c : [a,b] \to \mathbb{R}^2$, we obtain

$$\int_c \omega = \int_c df = f(x,y) - f(0,0) = x^2y^3 = x^8,$$

since $y = x^2$.

2. (a) Since ω is exact, we have $\omega = df = \sum_{j=1}^n \frac{\partial f}{\partial x_j}\,dx_j$, i.e., the coefficient functions of ω are $f_j = \frac{\partial f}{\partial x_j}$. Consequently, for $j,k \in \{1,\dots,n\}$, we have

$$\frac{\partial f_j}{\partial x_k} = \frac{\partial^2 f}{\partial x_k \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_k} = \frac{\partial f_k}{\partial x_j},$$

i.e., ω is closed.

(b) The coefficient functions of ω_0 are $f_1(x,y) = -\frac{y}{x^2+y^2}$ and $f_2(x,y) = \frac{x}{x^2+y^2}$. Then

$$\frac{\partial f_1}{\partial y} = -\frac{(x^2+y^2) - y\cdot 2y}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2} = \frac{(x^2+y^2) - x\cdot 2x}{(x^2+y^2)^2} = \frac{\partial f_2}{\partial x},$$

i.e., ω_0 is closed.

3. (a) Let $c(s) = (sx, sy, sz)$. Since f is homogeneous, we have $f(c(s)) = s^k f(x,y,z)$ and, therefore, on the one hand using the chain rule,

$$\frac{d}{ds}\Big|_{s=1} f(c(s)) = Df(c(1))(c'(1)) = \frac{\partial f}{\partial x}(x,y,z)x + \frac{\partial f}{\partial y}y + \frac{\partial f}{\partial z}z,$$

and on the other hand

$$\frac{d}{ds}\Big|_{s=1} s^k f(x,y,z) = kf(x,y,z).$$

Equating both sides yields the result.

(b) Note that, since μ is closed, we have $u_y = v_x$, $u_z = w_x$, $v_z = w_y$. A straighforward calculation yields

$$(k+1)\,df = (u + xu_x + yv_x + zw_x)\,dx + \\ (v + yv_y + xu_y + zw_y)\,dy + (w + zw_z + xu_z + yv_z)\,dz.$$

Using the above identities, we obtain

$$(k+1)\,df = (u + xu_x + yu_y + zu_z)\,dx + \\ (v + yv_y + xv_x + zv_z)\,dy + (w + zw_z + xw_x + yw_y)\,dz.$$

Since u, v, w are homogeneous of degree k, we conclude with (a),

$$(k+1)\,df = \\ (u+ku)\,dx + (v+kv)\,dy + (w+kw)\,dz = (k+1)(u\,dx + v\,dy + w\,dz).$$

Division by $k+1$ yields the result.

4. Let $f(x) = \frac{1}{2}\ln(\|x\|_2^2) = \frac{1}{2}\ln(\sum_j x_j^2)$. Then

$$df = \frac{1}{\|x\|_2^2}\sum_i x_i\,dx_i,$$

i.e., ω is exact. In the case $n = 3$, we have $f(x_1, x_2, x_3) = \ln\sqrt{x_1^2 + x_2^2 + x_3^2}$. This implies that

$$\int_c \omega = f(c(2k\pi)) - f(c(0)) = f(1, 0, 2k\pi) - f(1, 0, 0) = \ln\sqrt{1 + 4k^2\pi^2}.$$

5. The tangent vector $c'(t)$ is given by

$$c'(t) = (r'\cos\alpha - r\alpha'\sin\alpha, r'\sin\alpha + r\alpha'\cos\alpha).$$

This implies that

$$\begin{aligned} (\omega_0)_{c(t)}(c'(t)) &= -\frac{r\sin\alpha}{r^2}\,dx(c'(t)) + \frac{r\cos\alpha}{r^2}\,dy(c'(t)) \\ &= -\frac{\sin\alpha}{r}(r'\cos\alpha - r\alpha'\sin\alpha) + \frac{\cos\alpha}{r}(r'\sin\alpha + r\alpha'\cos\alpha) \\ &= \alpha'(\sin^2\alpha + \cos^2\alpha) = \alpha'(t). \end{aligned}$$

We finally obtain

$$\int_c \omega_0 = \int_0^1 (\omega_0)_{c(t)}(c'(t))\,dt = \int_0^1 \alpha'(t)\,dt = \alpha(1) - \alpha(0),$$

and

$$n(c) = \frac{1}{2\pi}(\alpha(1) - \alpha(0)) = \frac{1}{2\pi}\int_c \omega_0.$$