Analysis III/IV (Math 3011, Math 4201)

Solutions to Exercise Sheet 3 24.10.2011

1. (a) Assume that $x_n \to x$. Let $\epsilon > 0$ be given. Since f is continuous at x, there exists a $\delta 0$ such that $d_N(f(x), f(y)) < \epsilon$ for all $y \in M$ with $d_M(x, y) < \delta$. Since $x_n \to x$, there exists n_0 such that $d_M(x_n, x) < \delta$ for all $n \geq n_0$. Therefore, $d_N(f(x_n), f(x)) < \epsilon$ for all $n \geq n_0$. But this implies that $f(x_n) \to f(x)$.

   (b) We use sequential compactness. Let $y_n = f(x_n) \in f(K)$ with $x_n \in K$. Since K is compact, we have a subsequence $x_{n_j} \to x \in K$. Since f is continuous, we conclude from (a) that $y_{n_j} = f(x_{n_j}) \to f(x) \in f(K)$. But this means that y_n has a convergent subsequence in $f(K)$, i.e., $f(K)$ is compact.

   (c) Let $x \in f^{-1}(U)$ and $y = f(x) \in U$. Since U is open, there exists a $\epsilon > 0$ such that $U_\epsilon(y) \subset U$. Since f is continuous, there exists $\delta > 0$ such that $d_N(f(x), f(x')) < \epsilon$ for all $x' \in M$ with $d_M(x, x') < \delta$, i.e., $f(x') \in U_\epsilon(y)$ for all $x' \in U_\delta(x)$, i.e., $f(U_\delta(x)) \subset U_\epsilon(y) \subset U$, i.e., $U_\delta(x) \subset f^{-1}(U)$. This shows that $f^{-1}(U)$ is open.

   (d) Let $A \subset N$ be closed. Then $A^c = N \backslash A$ is open and $f^{-1}(A^c) \subset M$ is open, by (c). Since $f^{-1}(A^c) = (f^{-1}(A))^c$, we conclude that $f^{-1}(A) \subset M$ is closed.

2. Homework! Will be given in a later solution sheet.

3. Let a be the supremum of the set $\{f(x) \mid x \in M\}$. A priori, a can be infinity or a finite real number. There exists a sequence $x_n \in M$ such that $f(x_n)$ converges to the supremum. In case that the supremum is infinity, the sequence $f(x_n)$ becomes eventually larger than any positive number. Since M is compact, there exists a convergent subsequence $x_{n_j} \to x \in M$. Since f is continuous, we have $f(x_{n_j}) \to f(x)$. Since $f(x)$ is a well defined finite number and $f(x_{n_j})$ converges to the supremum, the supremum is a finite number and is attained at $x \in M$. Similar arguments hold for the infimum. This shows that f has a minimum and a maximum on the set M.

4. We have

$$\begin{aligned} \|v+w\|^2 + \|v-w\|^2 &= \langle v+w, v+w\rangle + \langle v-w, v-w\rangle \\ &= \|v\|^2 + \|w\|^2 + 2\langle v,w\rangle + \|v\|^2 + \|w\|^2 - 2\langle v,w\rangle \\ &= 2(\|v\|^2 + \|w\|^2). \end{aligned}$$

   Assume, w.l.o.g., $[a,b] = [0,1]$. Choose, e.g., $f(x) = 1$ for all $x \in [0,1]$ and $g(x) = x$ for all $x \in [0,1]$. Then $\|f\|_\infty = \|g\|_\infty = 1$, $\|f+g\|_\infty = 2$ and $\|f-g\|_\infty = 1$, contradicting to the paralellogram equation.

5. Let $v = \sum_i a_i e_i$. Then

$$\|v\| \leq \sum_i |a_i| \cdot \|e_i\|.$$

Let $M = \max\{\|e_1\|, \ldots, \|e_n\|\}$, then we obtain with Cauchy-Schwartz inequality $(\sum |x_i y_i| \leq (\sum |x_i|^2)^{1/2} (\sum |y_i|^2)^{1/2})$:

$$\|v\| \leq M \sum_i |a_i| \leq M\sqrt{n} (\sum_i |a_i|^2)^{1/2} = M\sqrt{n} \|v\|_2.$$

So we have $\| \cdot \| \leq C \| \cdot \|_2$ with $C = M\sqrt{n}$. Let $x_n \to x_0$ in the Euclidean metric, i.e., $\|x_n - x_0\|_2 \to 0$. This implies that $\|x_n - x_0\| \leq C \|x_n - x_0\|_2 \to 0$, as well and, therefore,

$$|\|x_n\| - \|x_0\|| \leq \|x_n - x_0\| \to 0.$$

This means that $\| \cdot \| : \mathbb{R}^n \to [0, \infty)$ is continuous with respect to the Euclidean metric. By Heine-Borel, S^{n-1} is closed and bounded with respect to the Euclidean metric, therefore compact. Every continuous function assumes its minimum and maximum on a compact set. Let min, max be the minimum and maximum of the map $\| \cdot \|$ on S^{n-1}. We must have $min > 0$, since $min = 0$ would mean that $\|x\| = 0$ for a vector in S^{n-1}, but $\|x\| = 0$ if and only if $x = 0$ (contradiction). We claim that

$$min \|v\|_2 \leq \|v\| \leq max \|v\|_2.$$

This is obviously true for $v = 0$. Let $v \neq 0$. Then $v / \|v\|_2 \in S^2$ and we have

$$min \leq \| \frac{v}{\|v\|_2} \| = \frac{\|v\|}{\|v\|_2} \leq max,$$

showing this inequality. But this inequality means that the two norms are equivalent.