Analysis III/IV (Math 3011, Math 4201)

Solutions to Exercise Sheet 1 10.10.2011

1. (a) Observe first that, for every non-zero rational number x, we have $\nu_p(x) > 0$. This immediately implies property (i) of a metric space. Property (ii) is obvious. The inequality

$$\nu_p(x+y) \le \max\{\nu_p(x), \nu_p(y)\} \le \nu_p(x) + \nu_p(y),$$

holds trivially if $x = 0$ or $y = 0$. Assume that $x, y \neq 0$ and $x = p^r \frac{a_0}{b_0}$, $y = p^s \frac{a_1}{b_1}$ with $r \ge s$ (otherwise, interchange x and y) and a_0, b_0, a_1, b_1 are not divisible by p. Then $x + y = p^s \frac{p^{r-s} a_0 b_1 + a_1 b_0}{b_0 b_1}$, and $b_0 b_1$ is not divisible by p. So we must have $x + y = p^{s'} \frac{c_0}{d_0}$ with $s' \ge s$ and c_0, d_0 not divisible by p, which implies

$$\nu_p(x+y) = p^{-s'} \le p^{-s} \le \max\{p^{-r}, p^{-s}\} \le p^{-r} + p^{-s} = \nu_p(x) + \nu_p(y).$$

Finally,

$$\begin{aligned} d_p(x,y) + d_p(y,z) = \nu_p(x-y) + \nu_p(y-z) &\ge \max\{d_p(x,y), d_p(y,z)\} \\ &\ge \nu_p((x-y)+(y-z)) = \nu_p(x-z) = d_p(x,z). \end{aligned}$$

(b) We have $d_p(x_n, 0) = \nu(p^n) = p^{-n} \to 0$.

(c) Assume that $n \ge m$. Then

$$d_p(x_n, x_m) = \nu_p\Big(\sum_{j=m}^{n-1} (a^{p^{j+1}} - a^{p^j})\Big) \le \max\{\nu_p(a^{p^{j+1}} - a^{p^j}) \mid m \le j \le n-1\}.$$

Since $a^{p^{j+1}} - a^{p^j} = a^{p^j}(a^{(p-1)p^j} - 1)$ and $\varphi(p^{j+1}) = (p-1)p^j$, we conclude from Euler's Theorem that $p^{j+1} | (a^{p^{j+1}} - a^{p^j})$. This implies that

$$d_p(x_n, x_m) \le p^{-(m+1)} \to 0$$

as $m \to \infty$. This shows that x_n is a Cauchy sequence.

(d) We know from Euler's Theorem that

$$a^{p^n} \equiv a^{p^{n-1}} \equiv a^{p^{n-2}} \equiv \cdots \equiv a \mod p.$$

$x_n \to \pm 1$ would mean that $x_n \pm 1 \to 0$ and, in particular $p | (a^{p^n} \pm 1)$ for n large enough. Together with $p | (a^{p^n} - a)$, this would imply that $p | (a \pm 1)$, contradicting to $2 \le a \le p-2$.
Using Euler's Theorem $a^{(p-1)p^n} \equiv 1 \mod p^{n+1}$ yields

$$p^{n+1} | (x_n^{p-1} - 1),$$

i.e., $d_p(x_n^{p-1}, 1) \le p^{-(n+1)} \to 0$.

2. Homework! Will be given in a later solution sheet.

3. (a) Fix an $\epsilon > 0$. Then there exists $n_0 \in \mathbb{N}$ such that, for $n, m \geq n_0$:

$$d(x_n, x_m) < \epsilon.$$

In particular, we have for all $n \geq n_0$:

$$d(x_{n_0}, x_n) < \epsilon.$$

Choose $x = x_{n_0}$ and $R = \max\{\epsilon, d(x, x_1), \ldots, d(x, x_{n_0-1})\}$. Then we have, obviously,

$$d(x, x_n) \leq R \qquad \text{for all } n \in \mathbb{N}.$$

(b) Since (x_n) and (y_n) are Cauchy sequences, they are bounded by (a), i.e., $|x_n|, |y_n| < C$ for $n \in \mathbb{N}$. Moreover, for given $\epsilon > 0$, there exist $n_0 \in \mathbb{N}$ such that

$$|x_n - x_m|, |y_n - y_m| < \frac{\epsilon}{2C} \qquad \text{for all } n, m \geq n_0.$$

This implies for $n, m \geq n_0$ that

$$\begin{aligned} |x_n y_n - x_m y_m| &= |x_n(y_n - y_m) + y_m(x_n - x_m)| \\ &\leq |x_n| \cdot |y_n - y_m| + |y_m| \cdot |x_n - x_m| < C\frac{\epsilon}{2C} + C\frac{\epsilon}{2C} = \epsilon, \end{aligned}$$

i.e., $(x_n y_n)$ is a Cauchy sequence.