Analysis III/IV (Math 3011, Math 4201)

Solutions to Exercise Sheet 9 17.12.2009

1. (a) The 2×3-matrix $Df(p)$ can never have rank 3, so all points are critical.

   (b) We have

$$Dg(x,y,z) = \begin{pmatrix} 4x & 2(y-1) & 0 \\ 0 & -z\sin(y) & \cos(y)-1 \end{pmatrix}$$

   If $y = 1$, then $\cos(y) - 1 \neq 0$ and we get a critical point if and only if $x = 0$.
   If $y \neq 1$, then $\det \begin{pmatrix} 2(y-1) & 0 \\ -z\sin(y) & \cos(y)-1 \end{pmatrix} = 2(y-1)(\cos(y)-1) = 0$ if and only if $\cos(y) = 1$, i.e. when $y = 2\pi n$ with $n \in \mathbb{Z}$. So the set of critical points is

$$C = \{(0,1,z) \in \mathbb{R}^3 \mid z \in \mathbb{R}\} \cup \{(x, 2\pi n, z) \in \mathbb{R}^3 \mid x, z \in \mathbb{R}^3, n \in \mathbb{Z}\}.$$

2. (a) Define $f : \mathbb{R}^3 \to \mathbb{R}$ by $f(x,y,z) = x^4 + y^2 + 2z^2$. Then $M = f^{-1}(\{4\})$. Now $Df(x,y,z) = \begin{pmatrix} 4x^3 & 2y & 4z \end{pmatrix}$. The only critical point is 0 which is not in M. So 4 is a regular value of f and M is a manifold by Theorem 5.7.

   (b) We have $Df(p) = \begin{pmatrix} -4 & 2 & 4 \end{pmatrix}$. By Theorem 5.10, $T_pM = \mathrm{Ker} Df(p)$, so

$$T_pM = \{(x,y,z) \in \mathbb{R}^3 \mid -2x + y + 2z = 0\}.$$

3. Define $g : \mathbb{R}^3 \to \mathbb{R}^2$ by $g(x,y,z) = ((x-1)^2 + y^2 - 5, y - z)$. Then $M = g^{-1}(\{(0,0)\})$ is closed. It is also bounded, as $(x-1)^2 + y^2 = 5$ implies that $|x|, |y| \leq 3$, and $z = y$ shows that if $(x,y,z) \in M$, then $|x|, |y|, |z| \leq 3$. By the Heine-Borel Theorem, M is compact. To see that M is a manifold, look at

$$Dg(x,y,z) = \begin{pmatrix} 2(x-1) & 2y & 0 \\ 0 & 1 & -1 \end{pmatrix}$$

   To get a critical point, we need $y = 0$ and $x = 1$. But then $g_1(x,y,z) = -5 \neq 0$, so $(0,0)$ is a regular value. By Theorem 5.7, M is a manifold. To get the critical points of f, we define

$$F(x,y,z,\lambda,\mu) = x^2 + y^2 + z + \lambda((x-1)^2 + y^2 - 5) + \mu(y - z)$$

The partial derivatives of F give rise to the following equations.

$$\begin{aligned} 2x + 2(x-1)\lambda &= 0 \\ 2y + 2y\lambda + \mu &= 0 \\ 1 - \mu &= 0 \\ (x-1)^2 + y^2 - 5 &= 0 \\ y - z &= 0. \end{aligned}$$

So $\mu = 1$ and $z = y$. Using equation 2, we get $\lambda = \frac{-1-2y}{2y}$, and using this in equation 1 gives

$$\begin{aligned} 2x + 2(x-1)\frac{-1-2y}{2y} &= \frac{2x2y - 2x - 2x2y + 2 + 4y}{2y} \\ &= \frac{4y - 2x + 2}{2y} \\ &= \frac{2y - x + 1}{y} \end{aligned}$$

and this is equal to 0 if and only if $x = 2y + 1$. Equation 4 is then $4y^2 + y^2 = 5$, which gives that $y = \pm 1$ and $x = 1 \pm 2$. The critical points are therefore $p_1 = (3, 1, 1)$ and $p_2 = (-1, -1, -1)$. Now $f(p_1) = 11$ and $f(p_2) = 1$, which gives the result.

4. (a) The distance is maximal if and only if the square of the distance is maximal, so we look for the point with maximal value of the function $f(x, y, z) = (x-1)^2 + (y-2)^2 + (z-3)^2$, where $(x, y, z) \in S^2$. Define

$$F(x, y, z, \lambda) = (x-1)^2 + (y-2)^2 + (z-3)^2 + \lambda(x^2 + y^2 + z^2 - 1).$$

Taking partial derivatives, we get the following equations.

$$\begin{aligned} 2(x-1) + 2\lambda x &= 0 \\ 2(y-2) + 2\lambda y &= 0 \\ 2(z-3) + 2\lambda z &= 0 \\ x^2 + y^2 + z^2 &= 1. \end{aligned}$$

This gives that

$$\begin{aligned} x &= \frac{1}{1+\lambda} \\ y &= \frac{2}{1+\lambda} \\ z &= \frac{3}{1+\lambda} \end{aligned}$$

which together with the last equation gives $(1+\lambda)^2 = 14$. Therefore the extremal points are

$$\left(\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}\right) \quad \text{and} \quad \left(-\frac{1}{\sqrt{14}}, -\frac{2}{\sqrt{14}}, -\frac{3}{\sqrt{14}}\right).$$

The second point has the bigger distance to $(1, 2, 3)$. As S^2 is compact, this point represents the point on the sphere with the maximal distance to $(1, 2, 3)$.

(b) Elementary geometry shows that the largest rectangles will have their sides parallel to the x- and y-axes, so if (x, y) denotes a corner on the ellipse, the other corners are given by $(-x, y)$, $(x, -y)$ and $(-x, -y)$. In particular half the perimeter is given by $2x + 2y$ (assuming that $x, y \geq 0$). Note that the ellipse is a 1-dimensional manifold. We therefore want to maximize $f(x, y) = 2x + 2y$ on the ellipse. Define

$$F(x, y, \lambda) \quad = \quad 2x + 2y + \lambda(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1).$$

The partial derivatives give the equations

$$\begin{aligned} 2 + \frac{2\lambda x}{a^2} &= 0 \\ 2 + \frac{2\lambda y}{b^2} &= 0 \\ \frac{x^2}{a^2} + \frac{y^2}{b^2} &= 1. \end{aligned}$$

Solving these equations, and noting that we want $x, y \geq 0$, we get $\lambda = -\sqrt{a^2 + b^2}$, and

$$\begin{aligned} x &= \frac{a^2}{\sqrt{a^2 + b^2}} \\ y &= \frac{b^2}{\sqrt{a^2 + b^2}} \end{aligned}$$

which represents corner points for the rectangle of greatest perimeter inside the ellipse.

5. (a) We want to find the minimum of $f(u, v) = \frac{1}{p}u^p + \frac{1}{q}v^q$ under the condition $u \cdot v = 1$, $u, v > 0$. Note that the condition describes a 1-dimensional submanifold of $\mathbb{R}^2$, the points are the graph of $g(u) = \frac{1}{u}$ for $u > 0$. Note that this is not a compact manifold, but if $u \to 0$ or $u \to \infty$, we get $f(u, 1/u) \to \infty$, so the minimum is taken somewhere in the middle. Define

$$F(u, v, \lambda) \quad = \quad \frac{1}{p}u^p + \frac{1}{q}v^q + \lambda(u \cdot v - 1).$$

This gives rise to the equations

$$\begin{aligned} u^{p-1} + \lambda v &= 0 \\ v^{q-1} + \lambda u &= 0 \\ u \cdot v &= 1. \end{aligned}$$

This implies $v^p = -1/\lambda = u^q$, which implies $u^{p+q} = 1$. We therefore get only one critical point for $u = v = 1$, which has to be the minimum for f by the discussion above. Also note that $f(1,1) = 1$, so

$$1 \leq f(u,v)$$

for all positive u, v with $uv = 1$.

(b) This inequality clearly holds if u or v are zero. So assume they are not and devide by uv. The inequality is then equivalent to

$$\begin{aligned} 1 &\leq \frac{1}{p}\frac{u^p}{uv} + \frac{1}{q}\frac{v^q}{uv} \\ 1 &\leq \frac{1}{p}\left(\frac{u}{(uv)^{1/p}}\right)^p + \frac{1}{q}\left(\frac{v}{(uv)^{1/q}}\right)^q. \end{aligned}$$

Note that $\frac{u}{(uv)^{1/p}} \cdot \frac{v}{(uv)^{1/q}} = 1$, so this inequality follows from part (a).

(c) With $u_j = \frac{|x_j|}{(\sum_{i=1}^n |x_i|^p)^{\frac{1}{p}}}$ and $v_j = \frac{|y_j|}{(\sum_{i=1}^n |y_i|^q)^{\frac{1}{q}}}$ we get from (b)

$$\begin{aligned} \sum_{j=1}^n u_j v_j &\leq \sum_{j=1}^n \frac{1}{p}u_j^p + \frac{1}{q}v_j^q \\ &= \frac{1}{p}\sum_{j=1}^n \frac{|x_j|^p}{\sum_{i=1}^n |x_i|^p} + \frac{1}{q}\sum_{j=1}^n \frac{|y_j|^q}{\sum_{i=1}^n |y_i|^p} \\ &= \frac{1}{p} + \frac{1}{q} = 1, \end{aligned}$$

so

$$\sum_{i=1}^n |x_i y_i| \leq \left(\sum_{i=1}^n |x_i|^p\right)^{\frac{1}{p}} \cdot \left(\sum_{i=1}^n |y_i|^q\right)^{\frac{1}{q}}$$

by multiplication.

(d) The difficult part is the triangle inequality. But writing

$$|x_i + y_i|^p = |x_i + y_i|^{p-1}|x_i + y_i| \leq |x_i + y_i|^{p-1}|x_i| + |x_i + y_i|^{p-1}|y_i|$$

we get

$$
\begin{aligned}
\|x+y\|_p^p &= \sum_{i=1}^{n} |x_i + y_i|^p \\
&= \sum_{i=1}^{n} |x_i + y_i|^{p-1} |x_i + y_i| \\
&\leq \sum_{i=1}^{n} |x_i + y_i|^{p-1} |x_i| + |x_i + y_i|^{p-1} |y_i| \\
&\leq \left(\sum_{i=1}^{n} |x_i + y_i|^{(p-1)q} \right)^{\frac{1}{q}} \cdot \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}} + \\
&\quad \left(\sum_{i=1}^{n} |x_i + y_i|^{(p-1)q} \right)^{\frac{1}{q}} \cdot \left(\sum_{i=1}^{n} |y_i|^p \right)^{\frac{1}{p}}
\end{aligned}
$$

where we used the Hölder inequality in the last step. Since $1-\frac{1}{q} = \frac{1}{p}$ and $(p-1)q = pq - q = p$, this is equivalent to

$$
\left(\sum_{i=1}^{n} |x_i + y_i|^p \right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} |y_i|^p \right)^{\frac{1}{p}}
$$

which is exactly the triangle inequality.