Analysis III/IV (Math 3011, Math 4201)

Solutions to Exercise Sheet 8 10.12.2009

1. First notice that $f(1,1,0,\pi/2,0) = (0,0,0)$. By the Implicit Function Theorem, we have to look at the 3×3-matrix $\left(\frac{\partial f_i}{\partial x_j}(p)\right)$ with $i,j = 1,2,3$. This matrix for general p is

$$\begin{pmatrix} 2x & -\cos(uv) & 2z \\ 2x & 2y & 4z \\ y & x & 1 \end{pmatrix}$$

and at $p = (1,1,0,\pi/2,0)$ we get

$$\begin{pmatrix} 2 & -1 & 0 \\ 2 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

which has determinant 6. By the Implicit Function Theorem, there is a function $\eta : U \to \mathbb{R}^3$ with $U \subset \mathbb{R}^2$ an open set containing $(\pi/2, 0)$, such that $f(\eta(u,v),u,v) = 0$. That is, the variables x,y,z can be expressed as functions depending on u,v, namely as the various coordinates of η.

2. Let $x \in \mathbb{R}^n$. Then

$$L_{(x,0),n} = \{(\lambda x, 1-\lambda) \mid \lambda \in \mathbb{R}\}$$

and $\|(\lambda x, 1-\lambda)\|_2^2 = 1$ is equivalent to $\lambda = 0$ or $\lambda = \frac{2}{1+\|x\|_2^2}$. The second equation leads to $\lambda = \frac{2}{1+\|x\|_2^2}$, which means that

$$\varphi_1(x) = \left(\frac{2x}{1+\|x\|_2^2}, \frac{\|x\|_2^2-1}{1+\|x\|_2^2}\right).$$

Similarly, we obtain

$$\varphi_2(x) = \left(\frac{2x}{1+\|x\|_2^2}, 1 - \frac{\|x\|_2^2}{\|x\|_2^2+1}\right).$$

Let $X = \frac{2x}{1+\|x\|_2^2}$ and $Z = \frac{\|x\|_2^2-1}{1+\|x\|_2^2} = 1 - \frac{2}{1+\|x\|_2^2}$. This implies that $X = (1-Z)x$ and $\varphi_2^{-1}(X,Z) = \frac{X}{1+Z}$. Consequently,

$$\varphi_2^{-1}\circ\varphi_1(x) = \varphi_2^{-1}\left(\frac{2x}{1+\|x\|_2^2}, \frac{\|x\|_2^2-1}{1+\|x\|_2^2}\right) = \frac{2x}{1+\|x\|_2^2}\cdot\frac{1+\|x\|_2^2}{2\|x\|_2^2} = \frac{x}{\|x\|_2^2}.$$

Moreover, we have

$$\frac{\partial}{\partial x_j}\frac{x_i}{\|x\|_2^2} = \frac{\delta_{ij}}{\|x\|_2^2} - \frac{2x_ix_j}{\|x\|^4}.$$

This implies that

$$D(\varphi_2^{-1} \circ \varphi_1)(x) = \frac{1}{\|x\|_2^2} \left(\mathrm{Id}_n - 2\frac{1}{\|x\|_2^2} x^\top x \right).$$

Remark: Geometrically, the matrix $\frac{1}{\|x\|^2} x^\top x$ describes a projection on the line $\mathbb{R}v$ and $\mathrm{Id}_n - 2\frac{1}{\|x\|_2^2} x \top x$ is a reflection in the hyperplane orthogonal to v. This geometric interpretation implies that $\frac{1}{\|x\|_2^2}\left(\mathrm{Id}_n - 2\frac{1}{\|x\|_2^2} x^\top x\right)$ is an invertible matrix with inverse $\|x\|_2^2 \left(\mathrm{Id}_n - 2\frac{1}{\|x\|_2^2} x^\top x\right)$.

3. Four coordinate patches covering M are given by

$$\begin{aligned}
\varphi_1 &: (0, 2\pi) \times [0, 2) \to M, \quad \varphi_1(u_1, u_2) = (\cos u_1, \sin u_1, u_2 - 1),\\
\varphi_2 &: (-\pi, \pi) \times [0, 2) \to M, \quad \varphi_1(u_1, u_2) = (\cos u_1, \sin u_1, u_2 - 1),\\
\varphi_3 &: (0, 2\pi) \times [0, 2) \to M, \quad \varphi_1(u_1, u_2) = (\cos u_1, \sin u_1, 1 - u_2),\\
\varphi_2 &: (-\pi, \pi) \times [0, 2) \to M, \quad \varphi_1(u_1, u_2) = (\cos u_1, \sin u_1, 1 - u_2).
\end{aligned}$$

It is straightforward to show that the functions φ_i are continuous and injective and that the images are open in M and that their union covers all of M. Because of

$$D\varphi_i(u_1, u_2) = \begin{pmatrix} -\sin u_1 & 0 \\ \cos u_1 & 0 \\ 0 & \pm 1 \end{pmatrix},$$

the matrices $D\varphi_i(u_1, u_2)$ have rank 2. The continuity of the argument functions $S^1 \{1\} \to (0, 2\pi)$, $e^{iu} \to u$, and $S^1 \{-1\} \to (\pi, \pi)$ from Complex Analysis implies that the inverses of φ_i are also continuous.

The boundary ∂M is given by the images

$$\varphi_1((0, 2\pi) \times \{0\} \cup \varphi_2((-\pi, \pi) \times \{0\} = \{x \in \mathbb{R}^3 \mid x_1^2 + x_2^2 = 1, x_3 = -1\}$$

and

$$\varphi_3((0, 2\pi) \times \{0\} \cup \varphi_4((-\pi, \pi) \times \{0\} = \{x \in \mathbb{R}^3 \mid x_1^2 + x_2^2 = 1, x_3 = 1\}.$$