**Analysis III/IV (Math 3011, Math 4201)**

**Solutions to Exercise Sheet 6** 26.11.2009

1. From $f(x+h)-f(x) = Ah = Df(x)h+R(h)$ we conclude that $Df(x) = A$ and $R = 0$. Moreover, from

$$g(x+h)-g(x) = \langle h, Ax\rangle + \langle x, Ah\rangle + \langle h, Ah\rangle = \langle (A+A^\top)x, h\rangle + \langle h, Ah\rangle,$$

we conclude $g(x+h)-g(x) = x^\top(A+A^\top)h + R(h)$ with $R(h) = \langle h, Ah\rangle$. Since

$$0 \le \lim_{h\to 0} \frac{\|R(h)\|_2}{\|h\|_2} \le \lim_{h\to 0} \frac{\|A\| \cdot \|h\|_2^2}{\|h\|_2} = 0,$$

the correction term $R(h)$ behaves the right way and we have $Df(x) = x^\top(A+A^\top)$.

2. $f \circ c(t) = c_0$ for all $t$ implies that $(f\circ c)'(t) = 0$ for all $t$. The chain rule yields

$$(f\circ c)'(t) = Df(c(t))\dot c(t) = \langle \nabla f(c(t)), \dot c(t)\rangle = 0 \quad \text{for all } t.$$

$\nabla f(x)$ is perpendicular to the tangent vector $\dot c(0)$ of any curve $c : (-\epsilon, \epsilon) \to U$ with $f \circ c$ constant and $c(0) = x$.

3. (a) We have

$$\begin{aligned}\operatorname{div}(fF)(x) = \sum_{i=1}^n \frac{\partial (fF_i)}{\partial x_i}(x) &= \sum_i \frac{\partial f}{\partial x_i}(x)F_i(x) + f(x)\frac{\partial F_i}{\partial x_i}(x) \\ &= \langle \nabla f(x), F(x)\rangle + f(x)\operatorname{div}F(x).\end{aligned}$$

   (b) Using the product rule, we obtain

$$\nabla(fg)(x) = \left(\frac{\partial(fg)}{\partial x_1}(x), \dots, \frac{\partial(fg)}{\partial x_n}(x\right) = f(x)\nabla g(x) + g(x)\nabla f(x).$$

   This implies with (a)

$$\Delta(fg) = \operatorname{div}\nabla(fg) = \operatorname{div}(f\nabla g) + \operatorname{div}(g\nabla f) = f\Delta g + 2\langle \nabla f, \nabla g\rangle + g\Delta f.$$

4. Note first that $\frac{\partial r}{\partial x_i}(x) = \frac{x_i}{\sqrt{x_1^2+\cdots+x_n^2}} = \frac{x_i}{r(x)}$. This immediately implies that $\nabla r(x) = \frac{1}{r(x)}x$. Let $f = f_0 \circ r$. Then, by the chain rule, $\frac{\partial f}{\partial x_i}(x) = f_0'(r(x))\frac{\partial r}{\partial x_i}(x)$. Thus, we have

$$\nabla f(x) = \frac{f_0'(r(x))}{r(x)}x.$$

The components $F_i$ of $F(x) = \frac{1}{r(x)}x$ have the partial derivative

$$\frac{F_i}{\partial x_i}(x) = -\frac{1}{r^2(x)}\frac{\partial r}{\partial x_i}(x) + \frac{1}{r(x)} = -\frac{x_i^2}{r^3(x)} + \frac{1}{r(x)}.$$

This implies that

$$\mathrm{div}F(x) = \sum_{i=1}^{n}\frac{\partial F_i}{\partial x_i}(x) = -\frac{1}{r^3(x)}\left(\sum_i x_i^2\right) + \frac{n}{r(x)} = \frac{n-1}{r(x)}.$$

Let $f = f_0 \circ r$. We already know that $\nabla f(x) = f_0'(r(x))F(x)$, and thus, by Exercise 3(a) and $\langle F(x), F(x)\rangle = 1$,

$$\begin{aligned}(\Delta f)(x) &= (\mathrm{div}(\nabla f))(x) = \langle \nabla(f_0' \circ r)(x), F(x)\rangle + f_0'(r(x))\mathrm{div}F(x) \\ &= \langle \frac{f_0''(r(x))}{r(x)}x, F(x)\rangle + f_0'(r(x))\frac{n-1}{r(x)} = f_0''(r(x)) + \frac{n-1}{r(x)}f_0'(r(x)).\end{aligned}$$

5. We have $Df(x_1, x_2) = \begin{pmatrix} 2x_1 & 0 \\ 1 & 3x_2^2 \end{pmatrix}$. If $f$ were locally invertible at $x = (0,0)$, $Df(0,0) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ would have to be an invertible matrix, which it isn't. Since $Df(1,1) = \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}$ is invertible, $f$ is locally invertible at $x = (1,1)$, by the Inverse Function Theorem. Moreover,

$$Df^{-1}(1,2) = Df^{-1}(f(x)) = (Df(x))^{-1} = \frac{1}{6}\begin{pmatrix} 3 & 0 \\ -1 & 2 \end{pmatrix}.$$