Analysis III/IV (Math 3011, Math 4201)

Solutions to Exercise Sheet 5 19.11.2009

1. We have $A^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, and so $A^{2k} = \begin{pmatrix} (-1)^k & 0 \\ 0 & (-1)^k \end{pmatrix}$ and $A^{2k+1} = \begin{pmatrix} (-1)^k & 0 \\ 0 & (-1)^k \end{pmatrix} \cdot \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & (-1)^{k+1} \\ (-1)^k & 0 \end{pmatrix}$.

   Therefore we get

$$\begin{aligned} e^{tA} &= \sum_{k=0}^{\infty}(-1)^k \frac{t^{2k}}{(2k)!} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \sum_{k=0}^{\infty} \frac{t^{2k+1}}{(2k+1)!} \begin{pmatrix} 0 & (-1)^{k+1} \\ (-1)^k & 0 \end{pmatrix} \\ &= \begin{pmatrix} \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k}}{(2k)!} & 0 \\ 0 & \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k}}{(2k)!} \end{pmatrix} + \begin{pmatrix} 0 & -\sum_{k=0}^{\infty} \frac{(-1)^k t^{2k+1}}{(2k+1)!} \\ \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k+1}}{(2k+1)!} & 0 \end{pmatrix} \\ &= \begin{pmatrix} \cos t & 0 \\ 0 & \cos t \end{pmatrix} + \begin{pmatrix} 0 & -\sin t \\ \sin t & 0 \end{pmatrix} \\ &= \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \end{aligned}$$

   So

$$e^{tA} \cdot \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \cdot \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \alpha \cos t - \beta \sin t \\ \alpha \sin t + \beta \cos t \end{pmatrix}$$

2. Note that

$$(B^{-1}AB)^k = (B^{-1}AB)(B^{-1}AB)\cdots(B^{-1}AB) = B^{-1}A^kB$$

   and therefore

$$\begin{aligned} e^{tB^{-1}AB} &= \sum_{k=0}^{\infty} \frac{t^k}{k!}(B^{-1}AB)^k \\ &= \sum_{k=0}^{\infty} \frac{t^k}{k!} B^{-1}A^kB \\ &= B^{-1}\left(\sum_{k=0}^{\infty} \frac{t^k}{k!} A^k\right) B \\ &= B^{-1}e^{tA}B. \end{aligned}$$

3. Assume $|F(x)-F(y)| \leq L \cdot |x-y|$ for all $x, y \in I$ with $I \subset \mathbb{R}$ non-negative and closed and $0 \in I$ and $F(0) = 0$. Then in particular $F(x) \leq L \cdot x$ for

$x \in I$. So if $x \in I$, we get

$$\begin{aligned} \sqrt{x} &\leq L \cdot x \\ \frac{1}{\sqrt{x}} &\leq L \\ \frac{1}{L^2} &\leq x, \end{aligned}$$

but the last line can be violated by choosing $0 < x < \frac{1}{L^2}$. Therefore F does not satisfy a Lipschitz condition near 0.

4. Choose $F(x) = |x|$, then F is not differentiable at $x = 0$, but

$$|F(x) - F(y)| \;=\; |\,|x| - |y|\,| \;\leq\; |x - y|$$

so F satisfies a Lipschitz condition with $L = 1$.

5. We have

$$\begin{aligned} \|F(x,t) - F(y,t)\|_2 &= \| \begin{pmatrix} t(x_2 - y_2) \\ t(y_1 - x_1) \end{pmatrix} \|_2 \\ &\leq |t| \cdot \sqrt{(x_1 - y_1)^2 + (x_2 - y - 2)^2} = |t| \cdot \|x - y\|_2. \end{aligned}$$

Choosing $\mathbb{R}^2 \times (-T, T) \subset \mathbb{R}^2 \times \mathbb{R}$, we see that F is Lipschitz continuous in the x-coordinate on this set with Lipschitz constant $L = T$.

We obtain

$$\begin{aligned} \beta_1(t) &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \int_0^t \begin{pmatrix} s \\ 0 \end{pmatrix} ds = \begin{pmatrix} t^2/2 \\ 1 \end{pmatrix}, \\ \beta_2(t) &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \int_0^t \begin{pmatrix} s \\ -s^3/2 \end{pmatrix} ds = \begin{pmatrix} t^2/2 \\ 1 - t^4/(2^2 \cdot 2) \end{pmatrix}, \\ \beta_3(t) &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \int_0^t \begin{pmatrix} s - s^5/2^3 \\ -s^3/2 \end{pmatrix} ds = \begin{pmatrix} t^2/2 - t^6/(2^3 \cdot 3!) \\ 1 - t^4/(2^2 \cdot 2) \end{pmatrix}. \end{aligned}$$

This leads to the guess $\beta_n(t) \to \alpha(t)$ with

$$\alpha(t) = \begin{pmatrix} \frac{t^2}{2} - \frac{1}{3!}\left(\frac{t^2}{2}\right)^3 + \frac{1}{5!}\left(\frac{t^2}{2}\right)^5 \mp \dots \\ 1 - \frac{1}{2!}\left(\frac{t^2}{2}\right)^2 + \frac{1}{4!}\left(\frac{t^2}{2}\right)^4 \mp \dots \end{pmatrix} = \begin{pmatrix} \sin(t^2/2) \\ \cos(t^2/2) \end{pmatrix}.$$

One easily checks that this is the solution of the given initial value problem.