Analysis III/IV (Math 3011, Math 4201)

Solutions to Exercise Sheet 4 12.11.2009

1. x_n is a Cauchy sequence: Note that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} < \infty$. For every *epsilon* > 0 there is an n_0 such that $\sum_{n=n_0}^{\infty} \frac{1}{n^2} < \epsilon$, and therefore, for $n, m \geq n_0$, $m \geq n$:

$$d(x_n, x_m) \leq d(x_n, x_{n+1}) + \cdots + d(x_{m-1}, x_m) \leq \sum_{i=n}^{m} \frac{1}{i^2} < \epsilon.$$

2. Assume that $f : M \to M'$ is continuous and $U \subset M'$ is open. Let $x \in f^{-1}(U)$. Since U is open, there exists an $\epsilon > 0$ such that $B(f(x), \epsilon) \subset U$. Since f is continuous, there exists a $\delta > 0$ such that $f(z) \in B(f(x), \epsilon)$ for all $z \in M$ with $d(z, x) < \delta$. But this means that $B(x, \delta) \subset f^{-1}(U)$. Therefore, $f^{-1}(U)$ is open.

   Assume $f : M \to M'$ satisfies $f^{-1}(U)$ open in M for all open $u \subset M'$. Let $x \in M$. We want to prove continuity of f at x. Given an $\epsilon > 0$, $B := B(f(x), \epsilon) \subset M'$ is open. Then $f^{-1}(B)$ is open in M and contains x. Therefore, there exists a $\delta > 0$ such that $B(x, \delta) \subset f^{-1}(B)$. But this means that $d'(f(y), f(x)) < \epsilon$ for all $y \in M$ with $d(y, x) < \delta$.

3. (a) Look at $g(x) = f(x) - x$. Then $g(a) \geq 0$ and $g(b) \leq 0$, so there must be a $x \in [a, b]$ with $g(x) = 0$. This implies $f(x) = x$.

   (b) Since $f'(x) < 1$ for all $x \in [a, b]$ and $|f'(x)|$ is continuous on $[a, b]$, it attains its maximum M on $[a, b]$, which must satisfy $M < 1$. Using the Mean Value Theorem, we obtain

$$|f(x) - f(y)| \leq |f'(\xi)| \cdot |x - y| \leq M \cdot |x - y|,$$

   for some ξ between x and y. This means that $f : [a, b] \to [a, b]$ is a contraction on the metric space $(M, d) = ([a, b], d(x, y) = |x - y|)$. The statement of the exercise is then just an application of the Contraction Mapping Principle.

   (c) Choose $f(x) = a + b - x$. Then $f'(x) = -1$. Choose, e.g. $x_0 = a$, then we have $x_n = b$ for all odd n and $x_n = a$ for all even n.

4. We have $F(x, t) = 2tx$ and

$$|F(x, t) - F(y, t)| = 2|t| \cdot |x - y|,$$

   and if we restrict t to a finite interval $(-C, C)$, we have Lipschitz continuity of F in the x variable with constant $L = 2C$. Let $\beta_0 \equiv c$. We obtain

$$
\begin{aligned}
\beta_1(t) &= c + \int_0^t 2scds = c + t^2 c, \\
\beta_2(t) &= c + \int_0^t 2s(c + s^2 c)ds = c + t^2 c + \frac{t^4}{2} c, \\
\beta_3(t) &= c + t^2 c + \frac{t^4}{2} c + \frac{t^6}{3!} c.
\end{aligned}
$$

This suggests that the (unique) solution might be $x(t) = ce^{t^2}$. A check shows: $\dot{x}(t) = 2tce^{t^2} = 2tx(t)$ and $x(0) = c$.