Analysis III/IV (Math 3011, Math 4201)

Solutions to Exercise Sheet 2 29.10.2009

1. (a) *There exists an* $\epsilon > 0$ *such that, for all* n_0 *there exists an* $n \geq n_0$ *with* $d(x_n, x) \geq \epsilon$, or shorter, *there exists an* $\epsilon > 0$ *and a strictly increasing sequence* n_j *of natural numbers with* $d(x_{n_j}, x) \geq \epsilon$.

   (b) *There exists an* $\epsilon > 0$ *such that, for all* n_0 *there exists an* $n \geq n_0$ *and* $x \in [a, b]$ *with* $|f_n(x) - f(x)| \geq \epsilon$, or shorter, *there exists an* $\epsilon > 0$, *a strictly increasing sequence* n_j *of natural numbers, and a sequence* $x_j \in [a, b]$ *such that* $|f_{n_j}(x_j) - f(x_j)| \geq \epsilon$.

2. Assume that $f_n \to f \in B([a, b])$. Let $x \in [a, b]$ and $\epsilon > 0$. Then there exists an n_0 such that $d(f_n, f) < \epsilon/3$ for all $n \geq n_0$. In particular, $d(f_{n_0}, f) < \epsilon/3$. Since f_{n_0} is continuous, there is a $\delta > 0$ such that $|f_{n_0}(y) - f_{n_0}(x)| < \epsilon/3$ for all y with $|y - x| < \delta$. This implies that

$$\begin{aligned} |f(x) - f(y)| &\leq |f(x) - f_{n_0}(x)| + |f_{n_0}(x) - f_{n_0}(y)| + |f_{n_0}(y) - f(y)| \\ &< d(f, f_{n_0}) + \epsilon/3 + d(f, f_{n_0}) < \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon, \end{aligned}$$

   for all y with $|y - x| < \delta$. This means that f is continuous at x.

3. Choose, e.g., $U_i = (-\frac{1}{i}, \frac{1}{i})$. Then $\bigcap_{i=1}^{\infty} U_i = \{0\}$ and this is not an open set since no open ball around 0 is completely contained in it.

   Note that every open ball $B(x, r) = (x - r, x + r)$ in $\mathbb{R}$ contains both rational and irrational points ($\mathbb{Q}$ is dense in $\mathbb{R}$). If $\mathbb{Q}$ were open, there were an open ball around 0 completely contained in $\mathbb{Q}$, which contradicts to the above observation. Likewise, if $\mathbb{Q}$ were closed, there were an open ball about $\pi \in \mathbb{Q}^c = \mathbb{R} \backslash \mathbb{Q}$ consisting only of irrational points which, again, is not possible.

4. We have

$$\begin{aligned} \|v + w\|^2 + \|v - w\|^2 &= \langle v + w, v + w \rangle + \langle v - w, v - w \rangle \\ &= \|v\|^2 + \|w\|^2 + 2\langle v, w \rangle + \|v\|^2 + \|w\|^2 - 2\langle v, w \rangle \\ &= 2\left(\|v\|^2 + \|w\|^2\right). \end{aligned}$$

   Assume, w.l.o.g., $[a, b] = [0, 1]$. Choose, e.g., $f(x) = 1$ for all $x \in [0, 1]$ and $g(x) = x$ for all $x \in [0, 1]$. Then $\|f\|_\infty = \|g\|_\infty = 1$, $\|f + g\|_\infty = 2$ and $\|f - g\|_\infty = 1$, contradicting to the parallelogram equation.

5. Choose, e.g., $I_n = (0, \frac{1}{n})$. Then $I_1 \supset I_2 \supset I_3 \supset \cdots$ and

$$\bigcap_{n=1}^{\infty} I_n = \{0\}.$$