Analysis III/IV (Math 3011, Math 4201)

Solutions to Exercise Sheet 16 18.03.2010

1. (a) We have

$$g^{(k)}(x) = \begin{cases} 0, & x \le 0, \\ \frac{p(x)}{q(x)} e^{-\frac{1}{x^2}}, & x > 0, \end{cases}$$

with appropriate polynomials p, q. Obviously,

$$\lim_{x \to 0} \frac{p(x)}{q(x)} e^{-\frac{1}{x^2}} = 0,$$

so g is a smooth function. Similarly, $h(x)$ is a smooth function, and so is the product $g(x-a)h(x-b)$. Since $g(x-a) = 0$ for all $x \le a$ and $f(x-b) = 0$ for all $x \ge b$, we have $f_{a,b}(x) = 0$ for $x \le a$ and $x \ge b$. Note that for $x \in (a,b)$ we have

$$f_{a,b} = e^{-\frac{1}{(x-a)^2}} e^{-\frac{1}{(x-b)^2}} > 0.$$

This proves (a).

(b) Note that for every $x \in [-10, 10]$, there is at least one function $F_i(x) > 0$. Hence, we have $F_1(x) + F_2(x) + F_3(x) > 0$ for all $x \in [-10, 10]$. Obviously, we have

$$f_1(x) + f_2(x) + f_3(x) = \frac{F_1(x) + F_2(x) + F_3(x)}{F_1(x) + F_2(x) + F_3(x)} = 1,$$

and all functions $f_i : [-10, 10] \to [0, 1]$ are smooth. One easily checks with (a) that their supports lie in U_1, U_2 and U_3.

2. K is a manifold with boundary, and if we choose the identity map as global parametrisation, the induced orientation on its boundary $E = \partial K$ is such that the outer normal unit vector field of K is positively oriented. By Stokes' Theorem, we then have

$$\int_K d\omega = \int_E \omega = 4\pi abc.$$

On the other hand, we have

$$\int_K d\omega = 3 \int_K dx \wedge dy \wedge dz = 3 \int_K 1 dx\, dy\, dz = 3\mathrm{vol}(K).$$

Putting both results together, we end up with

$$\mathrm{vol}(K) = \frac{4}{3}\pi abc.$$