Analysis III/IV (Math 3011, Math 4201)

Solutions to Exercise Sheet 15 18.03.2010

1. (a) We have
$$d\omega = 3\,dx \wedge dy \wedge dz.$$

(b) An almost coordinate chart of E is given by

$$\varphi : V := (0, 2\pi)\times(-\pi/2, \pi/2) \to U \subset E, \quad \varphi(\alpha, \beta) = (a \cos\alpha \cos\beta, b \sin\alpha \cos\beta, c \sin\beta).$$

The points, which are not reached by this parametrisation form a smooth curve connecting south and north pole of the ellipse. This is a set of measure zero. Then we have

$$\begin{aligned} w_2 := \frac{\partial\varphi}{\partial\alpha}(\alpha, \beta) &= (-a \sin\alpha \cos\beta, b \cos\alpha \cos\beta, 0)^\top, \\ w_3 := \frac{\partial\varphi}{\partial\beta}(\alpha, \beta) &= (-a \cos\alpha \sin\beta, -b \sin\alpha \sin\beta, c \cos\beta)^\top, \\ w_1 := \frac{\partial\varphi}{\partial\alpha}(\alpha, \beta) \times \frac{\partial\varphi}{\partial\beta}(\alpha, \beta) &= \left(bc \cos\alpha \cos^2\beta, ac \sin\alpha \cos^2\beta, ab \sin\alpha \cos\beta\right)^\top. \end{aligned}$$

By construction w_1, w_2, w_3 have the same orientation as e_1, e_2, e_3 and at $\varphi(\pi, 0) = (-a, 0, 0)$ we have $w_1 = (-bc, 0, 0)$, so that the outer unit normal vector is positively oriented with respect to the orientation induced by this coordinate chart.

We have
$$\int_E \omega = \int_U \omega = \int_V \varphi^*\omega,$$
and

$$\begin{aligned} \varphi^*\omega &= a \cos\alpha \cos\beta\, d(b \sin\alpha \cos\beta) \wedge\, d(c \sin\beta) - \\ &- b \sin\alpha \cos\beta\, d(a \cos\alpha \cos\beta) \wedge d(c \sin\beta) + c \sin\beta\, d(a \cos\alpha \cos\beta) \wedge d(b \sin\alpha \cos\beta) = \\ &= a \cos\alpha \cos\beta(b \cos\alpha \cos\beta d\alpha - b \sin\alpha \sin\beta d\beta) \wedge c \cos\beta d\beta - \\ &- b \sin\alpha \cos\beta(-a \sin\alpha \cos\beta d\alpha - a \cos\alpha \sin\beta d\beta) \wedge c \cos\beta d\beta + \\ &+ c \sin\beta(-a \sin\alpha \cos\beta d\alpha - a \cos\alpha \sin\beta d\beta) \wedge (b \cos\alpha \cos\beta d\alpha - b \sin\alpha \sin\beta d\beta) = \\ &= abc(\cos^2\alpha \cos^3\beta + \sin^2\alpha \cos^3\beta + \sin^2\alpha \sin^2\beta \cos\beta + \cos^2\alpha \sin^2\beta \cos\beta)\, d\alpha \wedge d\beta = \\ &= abc(\cos^3\beta + \sin^2\beta \cos\beta)\, d\alpha \wedge d\beta = abc \cos\beta\, d\alpha \wedge d\beta. \end{aligned}$$

Thus

$$\begin{aligned} \int_E \omega = \int_V \varphi^*\omega = \int_{(0,2\pi)} \int_{(-pi/2,\pi/2)} abc \cos\beta\, d\beta d\alpha = \\ = 2\pi abc \int_{(-\pi/2,\pi/2)} \cos\beta\, d\beta = 4\pi abc. \end{aligned}$$

2. (a) Let $x = (x_1, x_2, x_3)$. We have

$$\frac{\partial}{\partial x_i} \frac{x_i}{(x_1^2 + x_2^2 + x_3^2)^{3/2}} = \frac{\|x\|^3 - 3x_i^2 \|x\|}{\|x\|^6}.$$

This implies that

$$d\omega = \sum_{i=1}^{3} \frac{\partial}{\partial x_i} \left(\frac{\partial}{\partial x_i} \frac{x_i}{\|x\|^3} \right) dx_1 \wedge dx_2 \wedge dx_3 = \frac{3\|x\|^3 - 3\|x\|^3}{\|x\|^6} = 0,$$

i.e., $d\omega$ is closed.

(b) Let $v = (v_1, v_2, v_3)$ and $w = (w_1, w_2, w_3)$. Then

$$v \times w = (v_2 w_3 - v_3 w_2, v_3 w_1 - v_1 w_3, v_1 w_2 - v_2 w_1)$$

and

$$\begin{aligned} \langle v \times w, x \rangle &= x_1(v_2 w_3 - v_3 w_2) + x_2(v_3 w_1 - v_1 w_3) + x_3(v_1 w_2 - v_2 w_1) \\ &= x_1\, dx_2 \wedge dx_3(v, w) + x_2\, dx_3 \wedge dx_1(v, w) + x_3\, dx_1 \wedge dx_2(v, w), \end{aligned}$$

which immediately implies the equation in part (b).

(c) We choose the (almost global) coordinate chart $\Phi : (0, 2\pi) \times (-\frac{\pi}{2}, \frac{\pi}{2}) \to S_r(0)$, given by

$$\Phi(\alpha, \beta) = (r \cos\alpha \cos\beta, r \sin\alpha \cos\beta, r \sin\beta).$$

Then we have

$$\begin{aligned} \frac{\partial \Phi}{\partial \alpha}(\alpha, \beta) &= (-r \sin\alpha \cos\beta, r \cos\alpha \cos\beta, 0), \\ \frac{\partial \Phi}{\partial \beta}(\alpha, \beta) &= (-r \cos\alpha \sin\beta, -r \sin\alpha \sin\beta, r \cos\alpha). \end{aligned}$$

At the point $x = \Phi(0,0) = (r, 0, 0)$ we have $n(x) = (1, 0, 0) = e_1$, $\frac{\partial \Phi}{\partial \alpha}(0,0) = (0, 1, 0) = e_2$ and $\frac{\partial \Phi}{\partial \beta}(0,0) = (0, 0, 1) = e_3$, so the outward unit normal vector is positively oriented. Since

$$\begin{aligned} \Phi^* dx_1 &= -r \sin\alpha \cos\beta\, d\alpha - r \cos\alpha \sin\beta\, d\beta, \\ \Phi^* dx_2 &= r \cos\alpha \cos\beta\, d\alpha - r \sin\alpha \sin\beta\, d\beta, \\ \Phi^* dx_3 &= r \cos\beta\, d\beta, \end{aligned}$$

we obtain

$$\begin{aligned} \Phi^*(x_1 dx_2 \wedge dx_3) &= r \cos\alpha \cos\beta (r^2 \cos\alpha \cos^2\beta) d\alpha \wedge d\beta, \\ &= r^3 \cos^2\alpha \cos^3\beta\, d\alpha \wedge d\beta, \\ \Phi^*(x_2 dx_3 \wedge dx_1) &= r \sin\alpha \cos\beta (r^2 \sin\alpha \cos^2\beta) d\alpha \wedge d\beta, \\ &= r^3 \sin^2\alpha \cos^3\beta\, d\alpha \wedge d\beta, \\ \Phi^*(x_3 dx_1 \wedge dx_2) &= r \sin\beta (r^2 \sin^2\alpha \sin\beta \cos\beta + r^2 \cos^2\alpha \sin\beta \cos\beta)) d\alpha \wedge d\beta, \\ &= r^3 \sin^2\beta \cos\beta\, d\alpha \wedge d\beta. \end{aligned}$$

This implies

$$\Phi^*\omega = (\cos^3\beta + \sin^2\beta\cos\beta)d\alpha\wedge d\beta = \cos\beta\, d\alpha\wedge d\beta,$$

and we conclude that

$$\int_{S_r(0)} \omega = \int_0^{2\pi}\int_{-\pi/2}^{\pi/2} \cos\beta\, d\beta\, d\alpha = \int_0^{2\pi} 2\, d\alpha = 4\pi.$$