Analysis III/IV (Math 3011, Math 4201)

Solutions to Exercise Sheet 13 25.02.2010

1. Let $\varphi_1(r,\alpha,\beta) = r\cos\alpha\cos\beta$, $\varphi_2(r,\alpha,\beta) = r\sin\alpha\cos\beta$ and $\varphi_3(r,\alpha,\beta) = r\sin\beta)$. We have

$$\varphi^*(dy) = d\varphi_2 = \sin\alpha\cos\beta\,dr + r\cos\alpha\cos\beta\,d\alpha - r\sin\alpha\sin\beta\,d\beta,$$

and, consequently,

$$\begin{aligned}\varphi^*(xdy) &= r\cos\alpha\cos\beta(\sin\alpha\cos\beta\,dr + r\cos\alpha\cos\beta\,d\alpha - r\sin\alpha\sin\beta\,d\beta)\\ &= r\sin\alpha\cos\alpha\cos^2\beta\,dr + r^2\cos^2\alpha\cos^2\beta\,d\alpha - r^2\sin\alpha\cos\alpha\sin\beta\cos\beta\,d\beta.\end{aligned}$$

2. Let $\omega_1 \in \Omega^i(U)$. Since ω_2 is exact, it is also closed. We conclude that

$$d(\omega_1\wedge\omega_2) = d\omega_1\wedge\omega_2 + (-1)^i\omega_1\wedge d\omega_2 = 0.$$

Since ω_2 is exact, there exists a differential form $\eta \in \Omega(M)$ such that $\omega_2 = d\eta$. For exactness of $\omega_1\wedge\omega_2$, we have to find a differential form $\mu \in \Omega(M)$ such that

$$\omega_1\wedge\omega_2 = d\mu.$$

We choose $\mu = (-1)^i\omega_1\wedge\eta$. Then we obtain

$$d\mu = \underbrace{(-1)^i d\omega_1\wedge\eta}_{=0} + \omega_1\wedge d\eta = \omega_1\wedge\omega_2.$$

3. For each i, choose a countable set of rectangles $Q_1^i, Q_2^i, \ldots$ such that

$$A_i \subset \bigcup_j Q_j^i$$

and

$$\sum_j v(Q_j^i) < \frac{\epsilon}{2^i}.$$

Then the set of rectangles Q_j^i is a again countable, we have

$$\bigcup_i A_i \subset \bigcup_{i,j} Q_j^i$$

and

$$\sum_{i,j} v(Q_j^i) < \sum_{i=1}^\infty \frac{\epsilon}{2^i} = \epsilon.$$

This shows that $\bigcup_i A_i$ is also a set of measure zero.