Analysis III/IV (Math 3011, Math 4201)

Solutions to Exercise Sheet 12 18.02.2010

1. A quick solution goes as follows: Two alternating k-forms in $\mathbb{R}^n$ are the same if they coincide on tuples of basis vectors with increasing indices. In the case $k = n$, there is only one such tuple, namely, $(e_1, e_2, \ldots, e_n)$. Since

$$\det(e_1, \ldots, e_n) = 1,$$

we have $\alpha = \det$.

We also provide another (longer) proof without using the above fact: Note first for any sequence $1 \leq i_1, \ldots, i_n \leq n$ of indices, we have $\alpha(e_{i_1}, \ldots, e_{i_n}) = 0$ if at least two of the indices coincide. Hence, the only sequences of indices for which α doesn't vanish, are permutations $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$. We denote the set of all such permutations by $\mathcal{S}_n$. Since α is alternating, we have for every $\sigma \in \mathcal{S}_n$:

$$\alpha(e_{\sigma(1)}, \ldots, e_{\sigma(n)}) = \operatorname{sign} \sigma.$$

Now, let $v_i = \sum_j a_{ij} e_j$ for $1 \leq i \leq n$. Then we have

$$\begin{aligned}
\alpha(v_1, \ldots, v_n) &= \alpha(\sum_j a_{1j} e_j, \ldots, \sum_j a_{nj} e_j) \\
&= \sum_{1 \leq i_1, \ldots, i_n \leq n} a_{1i_1} \ldots a_{ni_n} \alpha(e_{i_1}, \ldots, e_{i_n}) \\
&= \sum_{\sigma \in \mathcal{S}_n} a_{1\sigma(1)} \ldots a_{n\sigma(n)} \alpha(e_{\sigma(1)}, \ldots, e_{\sigma(n)}) \\
&= \sum_{\sigma \in \mathcal{S}_n} \operatorname{sign}(\sigma) a_{1\sigma(1)} \ldots a_{n\sigma(n)} \\
&= \det \begin{pmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \ldots & a_{nn} \end{pmatrix} = \det(v_1, \ldots, v_n).
\end{aligned}$$

2. We have

$$\varphi \wedge \psi = (x\, dx - y\, dy) \wedge (z\, dx \wedge dy + x dy \wedge dz) = x^2\, dx \wedge dy \wedge dz$$

and

$$\phi \wedge \varphi \wedge \psi = \phi \wedge (\varphi \wedge \psi) = z\, dz \wedge (x^2\, dx \wedge dy \wedge dz) = 0.$$

The exterior differentials are

$$\begin{aligned}
d\varphi &= dx \wedge dx - dy \wedge dy = 0, \\
d\psi &= dz \wedge dx \wedge dy + dx \wedge dy \wedge dz = 2\, dx \wedge dy \wedge dz, \\
d\phi &= dz \wedge dz = 0.
\end{aligned}$$

3. (a) Let $f(t) = g(tx, ty, tz) = t^k g(x, y, z)$ for $t > 0$. Using the chain rule, we conclude that

$$f'(t) = x\frac{\partial g}{\partial x}(tx, ty, tz) + y\frac{\partial g}{\partial y}(tx, ty, tz) + z\frac{\partial g}{\partial z}(tx, ty, tz) = kt^{k-1}g(x, y, z).$$

This yields, at $t = 1$,

$$x\frac{\partial g}{\partial x}(x, y, z) + y\frac{\partial g}{\partial y}(x, y, z) + z\frac{\partial g}{\partial z}(x, y, z) = kg(x, y, z).$$

(b) For simplicity, let $f_x := \frac{\partial f}{\partial x}$. Then

$$d\omega = (b_x - a_y)dx \wedge dy + (c_x - a_z)dx \wedge dz + (c_y - b_z)dy \wedge dz.$$

Since $d\omega = 0$, we conclude that $b_x = a_y$, $c_x = a_z$, and $c_y = b_z$. Moreover, we obtain

$$\begin{aligned}(k+1)df &= (a + xa_x + yb_x + zc_x)dx + (b + yb_y + xa_y + zc_y)dy + (c + zc_z + xa_z + yb_z)dz \\ &= (a + xa_x + ya_y + za_z)dx + (b + yb_y + xb_x + zb_z)dy + (c + zc_z + xc_x + yc_y)dz.\end{aligned}$$

Applying Euler's relation to the homogeneous functions a, b, c, we obtain

$$(k+1)df = (a + ka)dx + (b + kb)dy + (c + kc)dz,$$

i.e.,

$$df = a\,dx + b\,dy + c\,dz = \omega.$$

(c) We first get

$$d\omega = (a_x + b_y + c_z)dx \wedge dy \wedge dz.$$

Next, we conclude that

$$\begin{aligned}(k+2)d\eta &= (-zb_y + c + yc_y + c + xc_x - za_x)dx \wedge dy + \\ &+ (b + zb_z - yc_z - ya_x + b + xb_x)dz \wedge dx + \\ &+ (-xc_z + a + za_z + a + ya_y - xb_y)dy \wedge dz \\ &= (2c + xc_x + yc_y + zc_z)dx \wedge dy + \\ &+ (2b + xb_x + yb_y + zb_z)dz \wedge dx + \\ &+ (2a + xa_x + ya_y + za_z)dy \wedge dz.\end{aligned}$$

Applying Euler's relation to the homogeneous functions a, b, c, we obtain

$$(k+2)d\eta = (2c + kc)dx \wedge dy + (2b + kb)dz \wedge dx + (2a + ka)dy \wedge dz,$$

i.e.,

$$d\eta = c\,dx \wedge dy + b\,dz \wedge dx + a\,dy \wedge dz = \omega.$$

4. We have

$$df = \sum \frac{\partial f}{\partial x_i}\, dx_i = \omega^1_{(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3})} = \omega^1_{\nabla f}$$

and

$$\begin{aligned} d\omega^1_F &= d(f_1\, dx_1 + f_2\, dx_2 + f_3\, dx_3) \\ &= (\frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2})dx_1 \wedge dx_2 + (\frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3})dx_2 \wedge dx_3 + (\frac{\partial f_1}{\partial x_3} - \frac{\partial f_3}{\partial x_1})dx_3 \wedge dx_1 \\ &= \omega^2_{\mathrm{curl}F} \end{aligned}$$

and

$$\begin{aligned} d\omega^2_F &= d(f_1\, dx_2 \wedge dx_3 + f_2\, dx_3 \wedge dx_1 + f_3\, dx_1 \wedge dx_2) \\ &= df_1 \wedge dx_2 \wedge dx_3 + dx_1 \wedge df_2 \wedge dx_3 + dx_1 \wedge dx_2 \wedge df_3 \\ &= \left(\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3}\right) dx_1 \wedge dx_2 \wedge dx_3 \\ &= \mathrm{div}F\, dx_1 \wedge dx_2 \wedge dx_3. \end{aligned}$$

Using the above identities, we obtain

$$d(df) = d\omega^1_{\nabla f} = \omega^2_{\mathrm{curl} \circ \nabla f}.$$

Since $d(df) = 0$, this implies that $\mathrm{curl} \circ \nabla f = 0$. Similarly, we obtain

$$d(d\omega^1_F) = d\omega^2_{\mathrm{curl}F} = \mathrm{div} \circ \mathrm{curl}F\, dx_1 \wedge dx_2 \wedge dx_3.$$

Since $d(d\omega) = 0$ for all ω, we conclude that

$$\mathrm{div} \circ \mathrm{curl}F = 0.$$