**Analysis III/IV (Math 3011, Math 4201)**

**Solutions to Exercise Sheet 10** 4.02.2010

1. Obviously, $\frac{\partial \varphi}{\partial u_1} \times \frac{\partial \varphi}{\partial u_2}(u) \in \mathbb{R}^3$ is perpendicular to the vectors $v_1 := \frac{\partial \varphi}{\partial u_1}(u) \in \mathbb{R}^3$ and $v_2 := \frac{\partial \varphi}{\partial u_2}(u) \in \mathbb{R}^3$. So it remains to show that $c'(0)$ is a linear combination of these vectors $v_1, v_2$. For this we observe that $c : (a,b) \to S$ with $c(0) = \varphi(u)$ is the image of a curve $\tilde{c} : (a,b) \to U$ under the map $\varphi$. (We simply choose $\tilde{c} := \varphi^{-1} \circ c$.) Moreover, we have $\tilde{c}(0) = u$. Denote the components of $\tilde{c}$ by $\tilde{c}_1, \tilde{c}_2$. By the chain rule, we have
$$c'(0) = (\varphi \circ \tilde{c})'(0) = \frac{\partial \varphi}{\partial u_1}(\tilde{c}(0))\tilde{c}_1'(0) + \frac{\partial \varphi}{\partial u_2}(\tilde{c}(0)\tilde{c}_2'(0) = \tilde{c}_1'(0)v_1 + \tilde{c}_2'(0)v_2.$$
Thus $c'(0) \in \operatorname{span}\{v_1, v_2\}$, which we wanted to show.

2. We have $c'(t) = (1 - \cos t, \sin t)$ and
$$\begin{aligned} \|c'(t)\|_2^2 &= 2 - 2\cos t = 2(1 - \cos(t/2 + t/2)) \\ &= 2(1 + \sin^2(t/2) - \cos^2(t/2)) = 4\sin^2(t/2). \end{aligned}$$
This implies that $\|c'(t)\|_2 = 2\sin(t/2)$ and
$$L(c) = \int_0^{2\pi} \|c'(t)\|_2 dt = \int_0^{2\pi} 2\sin(t/2)dt = 4\int_0^{\pi} \sin(s)ds = 8.$$

3. The coefficient functions of $\omega$ are $f_1(x,y) = -\frac{y}{x^2+y^2}$ and $f_2(x,y) = \frac{x}{x^2+y^2}$. We obtain
$$\begin{aligned} \frac{\partial f_1}{\partial y}(x,y) &= -\frac{(x^2+y^2) - 2y^2}{(x^2+y^2)^2} = \frac{y^2 - x^2}{(x^2+y^2)^2}, \\ \frac{\partial f_2}{\partial x}(x,y) &= \frac{(x^2+y^2) - 2x^2}{(x^2+y^2)^2} = \frac{y^2 - x^2}{(x^2+y^2)^2}. \end{aligned}$$
This implies closedness of $\omega$.
Since $c'(t) = (-r\sin t, r\cos t)$, we obtain
$$dx(c'(t)) = -r\sin t, \quad dy(c'(t)) = r\cos t.$$
This implies that
$$\int_c \omega = \int_0^{2\pi} \omega_{c(t)}(c'(t))dt = \int_0^{2\pi} -\frac{r\sin t}{r^2}\, dx(c'(t)) + \frac{r\cos t}{r^2}\, dy(c'(t))dt = \int_0^{2\pi} \sin^2 t + \cos^2 t\, dt = 2\pi.$$
If we had $\omega = df$, then Lemma 6.10 would tell us that
$$\int_c \omega = f(c(2\pi)) - f(c(0)) = f(r,0) - f(r,0) = 0.$$
This contradicts to $\int_c \omega = 2\pi$.

4. The coefficient functions of $\omega$ are $f_1(x,y) = 2xy^3$ and $f_2(x,y) = 3x^2y^2$. We obtain

$$\begin{aligned}\frac{\partial f_1}{\partial y}(x,y) &= 6xy^2,\\ \frac{\partial f_2}{\partial x}(x,y) &= 6xy^2.\end{aligned}$$

This implies closedness of $\omega$. Using Poincaré's Lemma, we conclude that $\omega$ is exact. Obviously, we have $\omega = df$ with $f(x,y) = x^2y^3$. Using Lemma 6.10, we conclude that

$$\int_c \omega = \int_c df = f(x,y) - f(0,0) = x^2y^3 = x^8 = y^4.$$