Analysis III/IV (Math 3011, Math 4201)

Exercise Sheet 9 10.12.2009

1. Determine the critical points of
   (a) $f : \mathbb{R}^2 \to \mathbb{R}^3$ given by $f(x, y) = (x^2, 2x + e^x \cos(y), xy \sin(xy))$.
   (b) $g : \mathbb{R}^3 \to \mathbb{R}^2$ given by $g(x, y, z) = (2x^2 + (y - 1)^2, z(\cos(y) - 1))$.

2. Let $M = \{(x, y, z) \in \mathbb{R}^3 \mid x^4 + y^2 + 2z^2 = 4\}$.
   (a) Show that M is a manifold.
   (b) For $p = (-1, 1, 1)$, determine the tangent space T_pM.

3. Show that

$$M = \{(x, y, z) \in \mathbb{R}^3 \mid (x - 1)^2 + y^2 = 5, y = z\}$$

   is a compact manifold and the extremal values of $f(x, y, z) = x^2 + y^2 + z$ on M are 11 and 1.

4. (a) Find the point of the sphere $x^2 + y^2 + z^2 = 1$ which is at the greatest distance from the point $(1, 2, 3) \in \mathbb{R}^3$.
   (b) Find the rectangle of greatest perimeter inscribed in the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

5. Let $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$.
   (a) Show that

$$1 \leq \frac{1}{p} u^p + \frac{1}{q} v^q$$

   for all positive numbers u, v with $u \cdot v = 1$.
   Hint: Lagrange multipliers.
   (b) Show that

$$uv \leq \frac{1}{p} u^p + \frac{1}{q} v^q$$

   for all $u, v \geq 0$.
   (c) (Hölder's Inequality) Let $x, y \in \mathbb{R}^n$. Show that

$$\sum_{i=1}^{n} |x_i y_i| \leq \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}} \cdot \left(\sum_{i=1}^{n} |y_i|^q \right)^{\frac{1}{q}}.$$

   Hint: Use $u = \frac{|x_j|}{(\sum_{i=1}^{n} |x_i|^p)^{\frac{1}{p}}}$.

(d) Let $p > 1$. Show that

$$\|x\|_p \quad = \quad \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}}$$

defines a norm on $\mathbb{R}^n$.

Hint: Write

$$|x_i + y_i|^p = |x_i + y_i|^{p-1}|x_i + y_i| \leq |x_i + y_i|^{p-1}|x_i| + |x_i + y_i|^{p-1}|y_i|$$

and note that $p + q = pq$.