Analysis III/IV (Math 3011, Math 4201)

Exercise Sheet 8 3.12.2009

1. Define $f : \mathbb{R}^5 \to \mathbb{R}^3$ by

$$f(x,y,z,u,v) \quad = \quad \begin{pmatrix} x^2 - y\cos(uv) + z^2 \\ x^2 + y^2 - \sin(uv) + 2z^2 - 2 \\ xy - \sin(u)\cdot\cos(v) + z \end{pmatrix}.$$

Show that near $p = (1,1,0,\pi/2,0)$ the variables x, y, z can be expressed as functions depending on u, v such that $f(x,y,z,u,v) = 0$.

2. The sphere $S^n = \{x \in \mathbb{R}^{n+1} \mid \|x\|_2 = 1\}$ is an n-dimensional manifold. (You don't need to show this!) Let $n = (0,\ldots,0,1), s = (0,\ldots,0,-1) \in S^n$. For any two points $p, q \in \mathbb{R}^{n+1}$, $p \neq q$, let $L_{pq} = \{\lambda p + (1-\lambda)q \mid \lambda \in \mathbb{R}\}$ be the straight Euclidean line through p and q. Two coordinate patches of S^n are given by

$$\varphi_1 : \mathbb{R}^n \to S^n, \quad \varphi_1(x) = L_{(x,0),n} \cap (S^n \backslash \{n\})$$

and

$$\varphi_2 : \mathbb{R}^n \to S^n, \quad \varphi_2(x) = L_{(x,0),s} \cap (S^n \backslash \{s\}).$$

(Again, you don't need to show this!) Calculate the images $\varphi_1(x_1,\ldots,x_n)$ and $\varphi_2(x_1,\ldots,x_n)$ explicitely as well as the coordinate change $\varphi_2^{-1} \circ \varphi_1 : \mathbb{R}^n\backslash\{0\} \to \mathbb{R}^n\backslash\{0\}$. Show that $D(\varphi_2^{-1} \circ \varphi_1)(x) = \frac{1}{\|x\|_2^2}\left(\mathrm{Id}_n - 2\frac{1}{\|x\|_2^2} x^\top x\right)$, where $x^\top x = (x_i x_j)_{1 \le i,j \le n}$.

Remark: The maps φ_1^{-1} and φ_2^{-1} are called *stereographic projections* and play an important role.

3. Let

$$M = \{x \in \mathbb{R}^3 \mid x_1^2 + x_2^2 = 1,\ -1 \le x_3 \le 1\}$$

be a cylinder. Show that M is a manifold with boundary and determine ∂M.