Analysis III/IV (Math 3011, Math 4201)

Exercise Sheet 7 26.11.2009

This set of homeworks should be handed in by Thursday, 3 December 2009 in the lecture.

1. Let $U \subset \mathbb{R}^n$ be open and $f : U \to \mathbb{R}^n$ be a C^k map with $k \geq 1$. Assume that $Df(x)$ is invertible for all $x \in U$. Use the Inverse Function Theorem to show that $f(U) \subset \mathbb{R}^n$ is open.

2. The proof of the Inverse Function Theorem contains an iteration procedure to calculate the preimage $f^{-1}(y)$ of a value y locally. Given the function $f(x) = x^2$, extract this procedure from the proof to obtain an iteration procedure to calculate $\sqrt{3}$ for nearby starting points and carry out the first three iterations for a (close enough) start value of your own choice.

3. Show that the set
$$M := \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid \left(\sqrt{x_1^2 + x_2^2} - 2\right)^2 + x_3^2 = 1$$
is a 2-dimensional manifold by finding coordinate patches for every point of M.

   Hint: This set can be generated by a rotation of the circle $\{x \in \mathbb{R}^3 \mid (x_1 - 2)^2 + x_3^2 = 1, x_2 = 0\}$ in the x_1, x_3-plane around the x_3-axis.

4. Let $U \subset \mathbb{R}^k$ be open and $f : U \to \mathbb{R}$ be smooth. Show that the graph of f is a k-dimensional manifold in $\mathbb{R}^{k+1}$.

5. A smooth curve $c : [0, a] \to \mathbb{R}^n$ is called *arc-length parametrised*, if we have
$$\|c'(t)\|_2 = 1 \quad \text{for all } t \in [0, a].$$
Given an arbitrary curve $c : [0, a] \to \mathbb{R}^n$ with $c'(t) \neq 0$ for all $t \in [0, a]$ and
$$L := \int_0^a \|c'(s)\|_2 ds.$$
(L can be interpreted as the length of the curve c.) Show that there exists a strictly increasing smooth map $\varphi : [0, L] \to [0, a]$ such that the new curve $\tilde{c} = c \circ \varphi : [0, L] \to \mathbb{R}^n$ is arc-length parametrised.

   Hint: Consider the strictly increasing function $L(t) = \int_0^t \|c'(s)\|_2 ds$ and define $\varphi = L^{-1}$.