Analysis III/IV (Math 3011, Math 4201)

Exercise Sheet 6 19.11.2009

1. (Easy Warmup!) Let $f : \mathbb{R}^n \to \mathbb{R}^n$, $f(x) = Ax + b$ with $A \in M_{n,n}(\mathbb{R})$ and $b \in \mathbb{R}^n$. Show that $Df(x) = A$ for all $x \in \mathbb{R}^n$. Let $g : \mathbb{R}^n \to \mathbb{R}$ defined by $g(x) = \langle x, Ax\rangle$, where $\langle\cdot,\cdot\rangle$ is the standard inner product of $\mathbb{R}^n$. Show that $Dg(x) = x^\top(A + A^\top)$.

2. Let $U \subset \mathbb{R}^n$ be open and $f : U \to \mathbb{R}$ be a smooth map. Let $c : [a, b] \to U$ be a smooth curve such that $f \circ c$ is constant. Show that

$$\langle \dot{c}(t), \nabla f(c(t))\rangle = 0.$$

   Interpret this fact geometrically: What can be said about the vector $\nabla f(x)$ in connection with curves through $x \in U$ which stay on a fixed level of the function f?

3. Let $U \subset \mathbb{R}^n$ be open and $F : U \to \mathbb{R}^n$ a vector field with component functions $F_1, \dots, F_n$. Then $\operatorname{div} F(x) = \sum_{i=1}^n \frac{\partial F_i}{\partial x_i}(x)$. The Laplacian of a function $f : U \to \mathbb{R}$ is defined as $\Delta f = \operatorname{div}(\nabla f)$. Show the following relations for functions $f, g : U \to \mathbb{R}$ and vector fields $X : U \to \mathbb{R}^n$:

   (a) $\operatorname{div}(fF) = \langle\nabla f, F\rangle + f \operatorname{div} F$

   (b) $\Delta(fg) = f\Delta g + 2\langle\nabla f, \nabla g\rangle + g\Delta f$

4. Let $r : \mathbb{R}^n \to \mathbb{R}$ be defined as $r(x) = \|x\|_2 = \sqrt{x_1^2 + \cdots + x_n^2}$. Show that for $x \in \mathbb{R}^n\backslash\{0\}$ we have

$$\nabla r(x) = \frac{1}{r(x)} x.$$

   Let $f_0 : (0, \infty) \to \mathbb{R}$ be a smooth map and $f : \mathbb{R}^n\backslash\{0\} \to \mathbb{R}$, $f = f_0 \circ r$. Show that

$$\nabla f(x) = \frac{f_0'(r(x))}{r(x)} x.$$

   Let $F : \mathbb{R}^n\backslash\{0\} \to \mathbb{R}^n$, $F(x) = \frac{1}{r(x)} x$. Show that

$$\operatorname{div} F(x) = \frac{n-1}{r(x)}.$$

   Use these facts to show for radial functions $f = f_0 \circ r$:

$$\Delta f(x) = f_0''(r(x)) + \frac{n-1}{r(x)} f_0'(r(x)).$$

5. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$f(x_1, x_2) = (x_1^2, x_1 + x_2^3).$$

   Is f locally invertible at $x = (0, 0)$? Give a reason why f is locally invertible at $x = (1, 1)$ and determine $Df^{-1}(f(x))$.