Analysis III/IV (Math 3011, Math 4201)

Exercise Sheet 5 12.11.2009

1. Let $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Show that

$$e^{tA} \cdot \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \alpha \cos t - \beta \sin t \\ \alpha \sin t + \beta \cos t \end{pmatrix}$$

   Hint: Write $e^{tA} = \sum_{k=0}^{\infty} \frac{t^{2k}}{(2k)!} A^{2k} + \sum_{k=0}^{\infty} \frac{t^{2k+1}}{(2k+1)!} A^{2k+1}$.

2. Let A and B be $n \times n$ matrices with B invertible. Show that

$$e^{tB^{-1}AB} = B^{-1} e^{tA} B.$$

3. Show that $F(x) = x^{\frac{1}{2}}$ does not satisfy a Lipschitz condition on any closed non-negative interval $I \subset \mathbb{R}$ containing 0.

4. Give an example of a continuous function $F : \mathbb{R} \to \mathbb{R}$ which is not differentiable, but satisfies $|F(x) - F(y)| \leq L \cdot |x - y|$ for some $L \geq 0$ and all $x, y \in \mathbb{R}$.

5. Consider the differential equation

$$\dot{\alpha}(t) = F(\alpha(t), t), \quad \alpha(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

   with $F : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$,

$$F(x, t) = \begin{pmatrix} tx_2 \\ -tx_1 \end{pmatrix}.$$

   Show that F is Lipschitz continuous in x in a neighbourhood of $(\begin{pmatrix} 0 \\ 1 \end{pmatrix}, 0) \in \mathbb{R}^2 \times \mathbb{R}$. Calculate the first three Picard-Lindelöf iterations $\beta_1, \beta_2, \beta_3$ of $\beta_0(t) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Can you guess the unique solution of this differential equation?