Analysis III/IV (Math 3011, Math 4201)

Exercise Sheet 4 5.11.2009

1. (Easy Warmup!) Let (M,d) be a complete metric space and x_n a sequence with $d(x_n, x_{n+1}) \leq \frac{1}{n^2}$. Show that x_n is convergent.

2. Let (M,d) and (M',d') be two metric spaces. Show that $f : M \to M'$ is continuous if and only if $f^{-1}(U) \subset M$ is open for all open $U \subset M'$.

3. Let $f : [a,b] \to [a,b]$ be a continuous function.

   (a) Use the Intermediate Value Theorem to show that f has a fixed point, i.e., a point $x \in [a,b]$ with $f(x) = x$.

   (b) Assume that $f \in C^1[a,b]$ and that $|f'(x)| < 1$ for all $x \in [a,b]$. Show that, for every $x_0 \in [a,b]$, the sequence x_n with $x_{n+1} = f(x_n)$ (for all n) is converging to a fixed point of f.

   (c) Find a function $f \in C^1[a,b]$ with $\|f'\|_\infty \leq 1$, such that there is a sequence x_n with $x_{n+1} = f(x_n)$ (for all n) which is non convergent.

4. Consider the one dimensional ordinary differential equation

$$\dot{x}(t) = 2tx(t).$$

   Apply Picard-Lindelöf iterations to determine the solution of this differential equation with the initial value $x(0) = c$, i.e., calculate the sequence $\beta_n : \mathbb{R} \to \mathbb{R}$, where $\beta_0 \equiv c$ and $\beta_{n+1} = T\beta_n$ for all $n \geq 0$ with T defined as in the lecture.

   β_n should converge to a particular power series. Do you see what function is represented by this power series?