Analysis III/IV (Math 3011, Math 4201)

Exercise Sheet 3 29.10.2009

1. Define $C^1[a,b]$ to be the sub-vector space of $C[a,b]$ consisting of those $f : [a,b] \to \mathbb{R}$, which have a continuous derivative $f' : [a,b] \to \mathbb{R}$ (here $f'(a) = \lim_{h \to 0^+} \frac{f(a+h)-f(a)}{h}$ is assumed to exist, and similarly with $f'(b)$).

   (a) Show that $\|\cdot\|_{C^1} : C^1[a,b] \to [0,\infty)$ defined by

   $$\|f\|_{C^1} \quad = \quad \sup\{|f(x)| \in \mathbb{R} \,|\, x \in [a,b]\} + \sup\{|f'(x)| \in \mathbb{R} \,|\, x \in [a,b]\}$$

   gives a norm on $C^1[a,b]$.

   (b) Show that $\|\cdot\|_* : C^1[a,b] \to [0,\infty)$ defined by

   $$\|f\|_* \quad = \quad \sup\{|f'(x)| \in \mathbb{R} \,|\, x \in [a,b]\}$$

   does not give a norm on $C^1[a,b]$.

   (c) Show that $D : C^1[a,b] \to C[a,b]$ given by $D(f) = f'$ is continuous, if we use $\|\cdot\|_{C^1}$ on $C^1[a,b]$ and the supremum norm $\|\cdot\|_\infty$ on $C[a,b]$.

2. (Warning: This exercise is a challenge!) Assume that $f_n \in C^1[a,b]$ and that there exists a pointwise limit $f(x) = \lim_{n\to\infty} f_n(x)$ for all $x \in [a,b]$. Assume also that there is a constant $C > 0$ such that

   $$|f_n'(x)| \leq C \quad \text{for all } n \text{ and } x \in [a,b].$$

   Show that f_n converges uniformly to f.

   Hint: Subdivide $[a,b]$ into small enough intervals on which the functions f_n don't vary too much (because of the bound on the derivatives) and then use simultaneous pointwise convergence at all end points of these small intervals.

   Remark: The upshot of the exercise is that if pointwise convergence of C^1-functions on a compact interval is not uniform, the derivatives of these functions have to explode.

3. We consider the following sequence $f_n \in C[0,1]$:

   $$f_n(x) = \begin{cases} n^{2/3} & \text{if } 0 \leq x \leq \frac{1}{n}, \\ 2n^{2/3} - n^{5/3}x & \text{if } \frac{1}{n} < x < \frac{2}{n}, \\ 0 & \text{if } \frac{2}{n} \leq x \leq 1. \end{cases}$$

   Let $\|f\|_1 = \int_0^1 |f(x)|dx$ and $\|f\|_2 = \left(\int_0^1 |f(x)|^2 dx\right)^{1/2}$. Show that $f_n \to 0$ with respect to $\|\cdot\|_1$, but f_n is not convergent with respect to $\|\cdot\|_2$.