Analysis III/IV (Math 3011, Math 4201)

Exercise Sheet 2 22.10.2009

This set of homeworks should be handed in by Thursday, 29 October 2009 in the lecture.

1. In this exercise you have to negate statements. For example: Let (X, d) be a metric space. The negation of $f : X \to \mathbb{R}$ is continuous at $x \in X$ is: There exists an $\epsilon > 0$ such that for all $\delta > 0$ there is a point $y \in X$ with $d(x, y) < \delta$ and $|f(x) - f(y)| \geq \epsilon$.

   (a) Let (X, d) be a metric space and $x_n \in X$ be a sequence. Negate the statement $x_n \to x$.

   (b) Let $f_n, f \in B([a, b])$. Negate the statement that f_n converges uniformly to f.

2. For $f, g \in B([a, b])$ let $d(f, g) := \sup_{x\in[a,b]} |f(x) - g(x)|$ and $C([a, b]) \subset B([a, b])$ be the subspace of continuous functions. Assume $f_n \in C([a, b])$ is a sequence with $f_n \to f \in B([a, b])$. Prove that the limit f is continuous.

   Hint: Use the (ϵ, δ)-definition of continuity and the fact that f_n is uniformly close to f, for large enough n, and that f_n is continuous. Combine all this to create a precise proof.

3. Find an example of infinitely many open sets $U_i \subset \mathbb{R}$ such that $\bigcap_{i=1}^{\infty} U_i$ is not open. Show that the set of rational numbers $\mathbb{Q} \subset \mathbb{R}$ is neither open nor closed in $\mathbb{R}$.

4. Let $\| \cdot \|$ be the norm of an inner product space $(V, \langle \cdot, \cdot \rangle)$. Show that the parallelogram equation is satisfied:

   $$\|v + w\|^2 + \|v - w\|^2 = 2(\|v\|^2 + \|w\|^2).$$

   Let $C([a, b])$ be equipped with the supremum norm $\|f\|_\infty = \sup_{x\in[a,b]} |f(x)|$. Show that the parallelogram equation is not satisfied and, hence, $d(f, g) = \|f - g\|_\infty$ is not induced from an inner product space.

5. Show that the statement of the *Nested Interval Principle* is no longer true if one replaces closed intervals by open intervals.