Analysis III/IV (Math 3011, Math 4201)

Exercise Sheet 16 11.03.2010

1. In this exercise, we construct a partition of unity for the closed interval $M = [-10, 10]$ (one dimensional manifold with boundary $\partial M = \{-10, 10\}$). Therefore, we introduce the functions

$$g(x) = \begin{cases} 0, & x \leq 0, \\ e^{-\frac{1}{x^2}}, & x > 0, \end{cases}, \qquad h(x) = \begin{cases} e^{-\frac{1}{x^2}}, & x < 0, \\ 0, & x \geq 0. \end{cases}$$

   (a) Let $a < b$. Show that $f_{a,b}(x) = g(x-a)h(x-b)$ is a smooth function, $f(x) > 0$ for $x \in (a, b)$ and $f(x) = 0$ for $x \leq a$ and $x \geq b$.

   (b) Let $F_1, F_2, F_3 : M \to [0, \infty)$ be defined as

$$F_1(x) = f_{-10,-3}(x), \quad F_2(x) = f_{-4,4}(x), \quad F_3(x) = f_{3,10}(x),$$

   and $f_i : M \to [0, 1]$, $f_i(x) = \frac{F_i(x)}{F_1(x)+F_2(x)+F_3(x)}$. Show that f_1, f_2, f_3 is a partition of unity, subordinated to the covering $U_1 = [-10, -2)$, $U_2 = (-5, 5)$ and $U_3 = (2, 10]$ of M.

2. This exercise is a continuation of an earlier exercise. Conclude from (a) and (b) in Exercise 1 of Sheet 15 that the volume of

$$K := \{(x, y, z) \mid \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1\}$$

   is

$$\operatorname{vol}(K) = \int_K 1 \, dx \, dy \, dz = \frac{4}{3}\pi abc.$$